Improved Estimation of Subsurface Magnetic Properties using Minimum Mean-Square Error Methods

Dr. scient thesis

by

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to Marianne & Rune Morten
Abstract

Aeromagnetic measurements have for many decades been used in frontier exploration for hydrocarbons, but the scarce information in the magnetic data sets have restricted their use elsewhere. Recently, new sensors and navigation systems have led to a considerable improvement of data quality, which enables the detection of even subtle variations in the magnetic properties encountered within the sedimentary succession in basins. The interpretation of magnetic data is, however, non-trivial due to their “non-uniqueness”, i.e., many subsurface configurations may give similar magnetic anomalies. The question of what can, and may be even more important, what cannot be determined from the magnetic measurements is of crucial importance. These questions are addressed very theoretically in the literature, and the published inversion methods are based on very simplified geometrical models of the subsurface. Some inversion methods provide estimates that are totally meaningless from a geological point of view.

Modern exploration for oil and gas is based heavily on the interpretation of seismic data prior to the drilling of exploration wells. At a certain stage in the exploration process, geological models are constructed based on seismic data and well information. The drilling of wells provides a complete lithological succession, and because the “normal” susceptibility variations for sedimentary rocks are fairly well established, it is possible to produce a “geological model of magnetic susceptibilities”. In order to interpret these complicated geological susceptibility models, an inversion method is proposed, based on constrained Minimum Mean-Square Error (MMSE) estimation. The MMSE method allows the incorporation of available prior information, i.e., the geometries of the rock bodies and their susceptibilities. There can be included uncertainties into the estimation process. The computation exploits the subtle information inherent in magnetic data sets in an optimal way in order to “tune” the initial susceptibility model. The MMSE method includes a statistical framework, that allows the computation of not only the estimated susceptibilities, given by the magnetic measurements, but also the associated reliabilities of these estimations. This allows the evaluation of the reliabilities in the estimates before any measurements are made; an option which can be useful for survey planning.

The MMSE estimation method has been tested on a synthetic data set in order to compare the effects of various prior information. When more information is given as input to the estimation, the estimated models become closer to the “true” model, and the reliabilities in their estimates are increased. In addition, the method was
evaluated using a real geological model from a North Sea oil field. The geological model is based on seismic data and well information, including susceptibilities representing the various lithologies present in the model. Given that the geometrical model is correct, the observed mismatch between the forward calculated magnetic anomalies and the measured anomalies, causes changes in the susceptibility model which may show features of interesting geological significance to the explorationists. Such magnetic anomalies may be due to small fractures and faults not detectable on seismic, or local geochemical changes due to the upward migration of water or hydrocarbons.
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Chapter 1

Introduction

"The 1960's were dominated by automatic computers. This introduced a new and important tool into the interpretation process. It also introduced a new kind of human being into the interpretation; people who dislike empirical, "non scientific" and approximate manual methods. They have to find something exact, something mathematically called "the best solution". They have, however, forgotten the fact that the inverse potential problem does not have any unique solution. This may bring us back to the darkness of the 1930's with interpretations showing large and small wrinkles in the basement, now camouflaged as exact solutions from powerful computers."

Knut Aam, 1973 [1].

Potential field data (magnetic and gravity) have for many decades been used for regional structural geological mapping in exploration for hydrocarbons. The low resolution of the potential field data and the non-uniqueness in the forward modelling have limited the use of such data for detailed geological mapping on the oilfield scale.

Magnetic data, which will be the focus in this work, has proven to be significant in the mapping of geological features, e.g., the structure of the basement or large faults. Magnetic data is inexpensive, easy to acquire over large areas, and it represents an independent data set (compared to seismic data). The improved quality of modern aeromagnetic data has a potential of detecting subtle magnetic variations within the sedimentary basins. For this reason an increased interest in the oil industry for using aeromagnetic data in the exploration process has emerged.

Magnetic measurements can only provide a limited amount of information regarding the subsurface geology in sedimentary basins. The "inversion" of potential field measurements to create a "true" pattern of susceptibility (magnetic) or density (gravity) variations, is recognised to be impossible since many different patterns lead to the same observed anomalies. For this reason the inversion problem is sometimes called the less ambitious term "interpretation" in place of "inversion", to suggest that any computed "inverse" should be treated with suspicion. The question of just what
can, and perhaps more importantly, cannot be determined from a given set of measurements is of great importance in geophysical prospecting and only a few articles have directly addressed this issue, often in a theoretical way. Other articles deal with how the “non-uniqueness” can be “overcome”, for example by taking a least-squares solution of some kind, or by some regularisation process. The resulting inversion algorithms provide estimates, even when the estimates in reality are meaningless in a geological context.

Indeed, the situation is worse than the observations of “non-uniqueness” suggest. In many geophysical inversion problems, even a modest goal, such as the estimation of a small number of parameters, simply cannot be carried out robustly. It is therefore concluded that the “inversion” or “interpretation” process should include not just the computing of some estimate of the parameters of interest, but also an estimate of the reliability of these estimates. In particular, the user should be warned when a certain measurement provides little new information about a parameter (which is not uncommon). While such a result is perhaps disappointing, it is more useful in practice than some “inversion” or “interpretation” method which delivers some estimate of the parameters that is, in fact, totally useless.

Today hydrocarbon exploration is performed by the interpretation of seismic surveys prior to the drilling of wells. In relatively frontier areas, the use of 2D seismic is most common. As certain targets are selected for detailed mapping, 3D seismic surveys become available. Both 2D and 3D interpreted seismic data can place constraints on the inversion of magnetic measurements; the layers and faults derived from the seismic data are chosen to describe the geometry of the prior model. The prior magnetic properties (the susceptibilities) are estimated from one of the following: 1) well-log information, 2) from general knowledge, 3) from assumptions.

The goal of this work is to develop a flexible inversion method which can incorporate detailed geological models and prior information regarding the susceptibilities of the rocks encountered in the model. Various models for implementing prior information are tested on synthetic data and later applied to a real data set from a North Sea oil field. It is focussed on the optimal use of magnetic measurements in order to tune a prior geologic model. Mismatch between forward modelled magnetic anomalies and the real aeromagnetic measurements, may initiate hypotheses of alternative geological interpretations, e.g., increased/reduced magnetisation in layers caused by the migration of water, gas or oil. To change the geometry of the model based on the inversion of aeromagnetic measurements alone, is too speculative because the seismic data usually contain more reliable information regarding the geological framework. The proposed inversion method assumes a linear relation between the measurements and the prior model, thus excluding the use of geometric parameters such as the shape of the layers and faults. Based on the assumption that the susceptibilities of all rock units have a Gaussian distribution, the sensor noise is Gaussian and the geometrical model is fixed, a linear inversion method gives the best solution. The estimator that minimises the Mean-square Error (MMSE-method) is chosen.

In the present work only 2D modelling is used, because it makes the prototyping of
1.1 Overview of thesis

Chapter 2 introduces some examples of the use of potential field data in geologic mapping. The properties of the earth's magnetism and magnetic measurements are discussed and some standard processing methods are introduced. The challenges dealing the non-unique data from magnetic surveys are introduced and some standard methods for dealing with this kind of data are discussed, as well as an introduction to the strategy chosen in this thesis.

Chapter 3 defines the Gaussian random vectors and linear models with measurement noise. The statistical framework for implementing prior knowledge is introduced as well as some examples of spatial correlations between the parameters to be estimated. The inversion is performed by applying the theory for MMSE estimation and incorporating prior information about the subsurface geological geometries based on seismic data. Thus the seismically derived geometries are limiting the domain of possible inversion solutions. The inversion routine also includes a calculation of the reliabilities of the estimates. Measures for evaluating this estimation are defined. An important spin-off of this method is the possibility of estimating the reliabilities of the estimates even before the measurements are performed. This method assumes Gaussian variables which seems reasonable from a geological point of view. It further assumes a linear relationship between the variables and measurements which are satisfied once the geometries of the model are fixed.

Chapter 4 describes the layout of the geometric model used; the model is divided into rectangular cells (pixels) with a user specified, but fixed, size. The forward modelling is based on summing up the contributions from each rectangle with constant magnetisations. Regions are defined as the set of pixels with equal and constant magnetic properties. The modelling of magnetic gradients within the regions, which might be caused by some geochemical processes, is implemented. The information inherent in magnetic data is focused by analysing the matrix which maps the geometric model to the measurements. Finally some examples of forward modelling demonstrate the detection limits for magnetic measurements, the problem of end effects in modelling and the shape of the anomaly due to the direction and the latitude of the measurements.

Chapter 5 introduces the use of MMSE for the interpretation of magnetic anomalies. Several models for implementing prior knowledge are shown, starting with defining an "exact" model as a reference. The amount and sophistication of prior information used in the models is increased, and the quality of the estimated models improves. The various models are compared and discussed. Examples with "wrong" prior mod-
els are used to demonstrate the "robustness" of the estimation method. At the end of this chapter some examples of the method in survey planning and evaluation are shown; measurement densities, measurements heights and sensor noise is addressed as well as calculating the variance of the estimates prior to measurements.

Chapter 6 includes a real case example from the North Sea. A geologic model based on interpreted seismic data is used to define the geometry of the layers. Magnetic properties from well-logs are assigned to the various geological entities. A profile from an airborne magnetic survey covering the same location as the seismic is used in the inversion. The initial mismatch between the measured and forward modelled anomalies is used to modify the interpretation of the geological model. The results show that magnetic measurements can be used to modify the prior geological model, as well as improving the reliabilities.

Concluding remarks and the directions for further work are finally presented in Chapter 7. This method, based on MMSE estimation, is a robust and flexible method for constraining the inversion by modelling and blending the available prior information with the magnetic measurements in an optimal fashion. Even though magnetic data contains only a small amount of information, it can contribute to an improved geological interpretation.

1.2 Contribution of thesis

The usefulness of magnetic data in petroleum exploration has been discussed for decades. Magnetic measurements generally do not contain enough information to perform a unique inversion calculation. For this reason magnetic data has mainly been used in iterative forward modelling. Current inversion methods are based on simple geometrical models (cylinders, rectangular prisms, etc.), or for the separation of two homogenous layers with a susceptibility contrast. The limited capability of incorporating detailed geological information, e.g., from seismic and well data have led to very little use of aeromagnetic data in petroleum exploration.

The goal of this work has been to develop an inversion method which, in a flexible manner, allows the incorporation of detailed prior information. The model may be represented by regions of pixels with gradients or fixed susceptibilities within each region; various options of correlation between pixels or regions may also be encoded. Thus, the input to the inversion process includes the geological geometries of the subsurface based on seismic data as well as general information regarding susceptibilities of the various lithologies encountered. In this way, even though the information inherent in magnetic data is limited, it may contribute substantially to an improved geological understanding, because it represents independent physical information.

The inversion method comprise a statistical framework which allows the computation of not only an estimate of the susceptibilities of the various formations in the prior geological model (given the aeromagnetic measurements), but also an associated
reliability of the estimates.

Another option in this inversion method is to calculate the reliabilities of an estimation even when no aeromagnetic data are available. The only information required are the prior uncertainties in the susceptibilities for each geological unit, the geometry of the model and the noise level in the sensor. This option may also be useful for the planning of a new aeromagetic survey.

It is proposed that this inversion method should be interfaced to standard seismic interpretation tools in order to effectively incorporate the geometries of the geological model. The prior magnetic susceptibilities of the subsurface units could easily be selected from in house databases.

The proposed inversion method gives the explorationist the option to use the additional information inherent in potential field data for an improved geological interpretation.
Chapter 2

Magnetic data and interpretation

2.1 Potential field data in exploration

Potential field data, like magnetics and gravity, has for decades been used in general geological mapping and exploration for minerals and hydrocarbons. Use of magnetic data in geologic mapping is the oldest discipline in geophysics. The paper “The examination of Iron Ore deposits by Magnetic measurements” by Thalen in 1879, was the first demonstration of using magnetic data for geologic mapping [73]. Magnetic data is still widely used in mapping mineral deposits since many minerals are associated with magnetic rock units [30, 69, 34].

Potential field data is also widely used for general geological mapping covering vast areas. Gunn et al [35] shows how aeromagnetic data can be used for mapping lithologies and structures onshore Australia, even in areas with limited outcrops. Skilbrei [67] has used aeromagnetic data to interpret the depth to the magnetic basement in the northern Barents Sea with success. By using advanced image processing routines, gravity data has been used to map Palaeozoic structures in the Barents Sea giving additional information to the seismic data characterising basin configurations [64]. Johansen et al [44] has used gravity data in addition to seismic and well data in mapping the continuation of the Caledonides north of Norway for an improved interpretation.

Integrated use of gravity and magnetic data has also proven successful in developing structural models where the geological units are poorly exposed onshore [43]. In offshore mapping, combined use of gravity and magnetic data has been shown useful for mapping the patterns of basement structure [19]. Gunn [33] has demonstrated the use of integrated interpretation of regional gravity and magnetic data to map sedimentary basins. Piskarev and Tchernyshev [60] have studied the features of gravity and magnetic fields in the vicinity of oil and gas reservoirs in West Siberia. These revealed links may be important, e.g., when mapping the poorly explored territories in Russia’s Arctic shelf.

High resolution magnetic data has been available the last five years. While older
magnetic surveys could be used to map the main structural trends and define the sedimentary basins which is of main interest in petroleum exploration, the high resolution magnetic data can even be used to map fine structural and sedimentary details within the basins [31]. Work by Fichler et al [20] shows that aeromagnetic anomalies may be useful in mapping sedimentary lithologies and even demonstrates correlation between gas seepage relates features and magnetic anomalies.

In this thesis the focus on the interpretation of magnetic data and it start with a description of the basic physics of magnetic measurements, followed by a presentation of current methods, problems and challenges.

2.2 Magnetic measurements and processing

The classical magnetic theory is similar to the electrical and the gravity theory; its basic concept is that point magnetic poles are analogous to point electrical charges and point masses, with a similar inverse-square law for the forces between the poles, charges and masses [73]. Magnetic units in the cgs and emu system are based on the concept that point magnetic poles are analogous to point electrical charges and point masses (gravity). The SI-units are based on the basic unit, the dipole and is explained as follows.

A magnetic dipole can be considered as two poles of strength \(+p\) and \(-p\) separated by a distance, \(\Delta l = 2l\), where \(l\) is the length from the centre to the end of the dipole. The magnetic dipole moment, \(m\) is a vector in the same direction as a unit vector, \(r_1\) (that extends from the negative pole toward the positive pole):

\[
m = 2lpr_1
\]

The magnetic field is caused by an electrical current. A current \(I\) creates a magnetisation field \(\Delta H\) at point \(P\), see figure 2.1 (from [73, p63]). According to Ampere’s law:

\[
\Delta H = (I\Delta l) \times \frac{r_1}{4\pi r^2}
\]

A body which is magnetisable will be magnetised by induction if it is placed within an external magnetic field. This magnetisation is measured by the magnetic polarisation, \(M\) and is called magnetic intensity or dipole moment per unit volume. For low magnetic fields, \(M\) is proportional to \(H\) and in the same direction:

\[
M = \kappa \cdot H
\]

(2.1)

where \(\kappa\) (the susceptibility) is a constant for each body indicating the ability to be magnetised inside an external field. The linearity in equation 2.1 holds only for weak magnetic fields.
The magnetic induction or magnetic flux, \( B \), is the total field including the effect of magnetisation and is related to the magnetic field by:

\[
B = \mu_0(H + \kappa \cdot H)
\]

where \( \mu_0 \) is the permeability of free space. The susceptibility is defined by:

\[
\kappa = \frac{\mu}{\mu_0} - 1
\]

where \( \mu \) is the permeability of the body (medium).

When the external magnetic field, \( H \), is decreased, the curve does not retrace the same path, but it shows a positive value of \( B \) when \( H = 0 \), see figure 2.2 (from [73, p64]) . This field is called remanent or residual magnetisation. A body’s ability to keep this residual is called coercive force. Heating a magnetised body leads to an increased susceptibility. Above this temperature, called the Curie point, at 550\(^\circ\)C, a body will lose its remanent magnetism.

The geomagnetic field (electromagnetic interaction) of the Earth consist of three parts:

- the main field has an internal origin from the core of earth and is in the range of 30000-60000 nT with relatively long wavelength variations,

- a smaller field, the external magnetic field, 0-2000 nT is mainly caused by solar activities,
• local magnetic anomalies, *crustal field* is caused by local changes in magnetisation of near-surface rocks.

Figure 2.3 (from [1]) illustrates the main field of the Earth and b shows typical anomaly-values for different places on the Earth. The *inclination* is defined as the angle of the main magnetic field measured at a certain point on the surface, whereas the *declination* is the angle between true geographic north and the magnetic north, see figure 2.4 (from [73, p68]). The *azimuth* defines the measurement direction.

The measured magnetic anomalies represent a combination of the main field and the crustal field. Figure 2.5 shows how the observed anomalies are generated, by the main field and an induced magnetic body in the crust.

The external magnetic field varies relatively rapid and the most severe effect is caused by magnetic storms and has an amplitude of up to 1000 nT. These effects have to be corrected by using base-station magnetometers.

Local magnetic anomalies are the focus in geological mapping by magnetic measurements. These magnetic anomalies are caused by the change in susceptibilities for different rocks and layers. Some rocks [14] are *diamagnetic* (quartz, calcite), i.e., they have slightly negative susceptibilities. Rocks that contains ferromagnetic minerals are called *ferromagnetic* and may have a relatively high susceptibility, e.g. diorite (10 – 100 × 10⁻³ SI – units). *Paramagnetic* minerals (olivin, pyroxenes and pure ilmenite) have weak positive susceptibilities. Rocks containing such minerals
Figure 2.3: a) A cross-section through the Earth and the main magnetic field, b) shows the global main magnetic field as a contour-map (in nT).
Figure 2.4: Elements of the Earth's magnetic field: $F_e$ is the magnitude of the field, I the inclination and D the declination.

are magnetic with a typical susceptibility in the range $0 - 0.1 \times 10^{-3}$ SI units. Imature sandstones are paramagnetic while mature sandstones are diamagnetic. For many decades these rocks were of no importance in geologic mapping due to their low magnetic influence on the measurements, but in recent times, with enhanced sensitivity in the measurements, their subtle anomalies may be detectable. Table A.1 on page 112 (from [73, p74]) lists susceptibilities for various minerals.

The magnetisation of rocks is remanent if it carries a permanent magnetic field caused by an previous ambient field. The Koenigsberger ratio, $Q$, is defined as the ratio between remanent and induced magnetisation. In some areas this ratio may have values in the range $10 - 100$ and thus represent a significant component of the total magnetisation. In most cases, however, the induced magnetisation is the dominant part. The measurements of remanent magnetisation is very expensive since it needs samples collected in situ, so numerical modelling normally makes the assumption of induced magnetisation.

Airborne measurements and preprocessing

Airborne magnetic measurements are widely used in offshore hydrocarbon exploration. The Geological survey of Norway has since the 1960's systematically mapped the Norwegian coastline. These regional studies were presented as contour maps with contour spacing of $20\gamma$ ($1\gamma = 1$ nT). In the 1980's and 1990's many companies have been performing semi-regional or local surveys, mainly for the purpose of hydro-
Figure 2.5: The observed field is a combination of the primary field (main field) caused by the “dipole Earth” and the secondary local field caused by an induced magnetised body.
carbon exploration. Mapping with gravimetric and magnetometric data is typically used in the early stages of exploration, as a tool for mapping the main structures and faults. This may give important information when planning a seismic survey, since the seismic image will be improved when measuring perpendicular to the strike of the main geological features. As the sensors and the navigation has dramatically improved over the last decade, it is even possible to use these kind of data for detailed mapping together with the interpretation of seismic data.

The current industry standard airborne magnetometer sensors are the pumped caesium and the helium sensors [40]. These sensors very accurate; manufacturers claims down to 0.001 nT, but for practical use 0.1 – 0.01 nT is realistic. The sampling rate of data will depend on the goal of the survey, but one sampling each 0.1 – 1 second along the profile is commonly used.

To correct for magnetic field due to non-geological features, like diurnal effects, a fixed base magnetometer is used. The daily variations are registered digital and used to correct the airborne measurements. If this noise is above some threshold, the survey may even be cancelled for a period of time.

The aeromagnetic flight lines will normally be parallel with line spacing of 500 to 2000 meters, depending of the aims of the survey. To correct for long wave changes during the measurements, some crossbones or tielines will be measured, see figure 2.6. The planning of the flight directions is important because the quality of the maps will be improved when measuring perpendicular to the main strike of the area. Altitude should be constant throughout the whole survey offshore; typically 100 – 200 meters above the sea’s surface.

The pre-processing of magnetic data is an important part of an aeromagnetic survey.
This is performed prior to the further processing, enhancement and interpretation of the data. The pre-processing consists of several steps [50]:

- verifying and editing the raw data,
- locating the data,
- parallax corrections,
- removing diurnals,
- removing the Earth’s regional field component,
- levelling the data,
- gridding and contouring

Many commercial software packages are available for this work but the knowledge of the persons in charge is important for good results.

**Processing**

Profiles with magnetic measurements will, after the pre-processing steps, normally be gridded for the following processing and image enhancements prior to the interpretation. The gridded data set is a 2D image and different processing algorithms and visualisation techniques can be applied for mapping structures, faults and even sedimentary basins. A variety of filters and transformations may be applied, some of them will convert fields representing physically measurable parameters into other fields representing different physically measurable parameters, some will enhance the data for visualisation and interpretation [54].

The wavelength and amplitude of magnetic anomalies are highly dependent on the altitude of the sensor relative to the subsurface. The position and the shape of the anomalies caused by a certain body is highly dependant on the inclination of the inducing magnetic field, see figure 4.5. The Fourier transform [73, p738] is particularly useful in processing magnetic data for a variety of operations [73, p107]:

- resolution of anomalies by downward or upward continuation
- changing the effective field inclination, e.g. Reduction-to-pole (RTP)
- calculations of derivatives
- general filtering (highpass, lowpass, bandpass)
- for magnetic modelling


**Figure 2.7:** The steps in a traditional forward modelling: based on proposed geometries and assigned susceptibilities, the theoretically magnetic anomalies are calculated and compared to the true magnetic measurements.

The upward/downward continuation is often used when integrating surveys with different surveys heights. The RTP is useful, especially for surveys at low inclination angle since the anomalies will be body-centered and positive/negative anomalies are spatially coincident with positive/negative susceptibilities for the materials measured.

### 2.3 Modelling methods

There are two main directions in modelling magnetic measurements. In *forward modelling*, the magnetic anomalies are calculated based on assumed geometries of the causal bodies. These anomalies are then compared with real magnetic measurements, and the mismatch between forward modelled and measured anomalies can then be used to change the geologic model in an iterative manner, see figure 2.7.

*Inverse modelling* is based on the automated "construction" of the geological model by inversion routines. If the causative model is complicated or detailed, most inversion methods need constraints regarding the magnetic properties and/or geometries to produce a reasonable geological model, see figure 2.8.
2.3. MODELLING METHODS

![Diagram](image)

**Figure 2.8:** The steps in traditional magnetic inversion: geological/geometrical constraints are encoded into the inversion, resulting in a proposed geological model.

### 2.3.1 Forward modelling

Talwani & Heirzler [72] introduced a method for the computation of a magnetic field caused by a 2D body with an arbitrary shape (infinite strike), named 2D modelling. Computation of the magnetic effect of arbitrarily shaped 3D bodies can be performed by approximating them to horizontal polygonal laminas [71] or by approximating them to a series of triangular facets [48]. A method based on Fast Fourier Transform for rapid computation of gravity and magnetic anomalies was developed by Bhattacharyya and Navolio [9].

Many authors [66, 13] have contributed in the development of methods for calculating the magnetic effect of a body with polygonal cross-section and limited strike, called 2.5D modelling. Coggon [15] introduced a method commonly named 2.75D modelling which calculates the magnetic field of a body of arbitrary constant cross-section on a profile perpendicular to the body at any length of the body.

As an example of a successful application of forward modelling, a publication by Valenta et al is referred [74]. In this work, forward modelling of a measured magnetic anomaly field with constraints from magnetic susceptibility data was performed to generate and test a three-dimensional model of the structural history of the Duchess area in Australia.

Interactive 3D modelling of gravity and magnetic data using interactive computer graphics, enables the interpreter to change both the geometry of the model as well as the density/susceptibility of the bodies in an effective manner [24]. This capability has shown to be very useful allowing the interpreter to perform the necessary changes
in the model to achieve a good fit between the measured and forward calculated anomalies [75].

### 2.3.2 Inverse modelling

A step forward in the modelling of magnetic data, is the use of automated inversion routines for the generation of geological models based upon measurements. Inversion methods are divided into two main directions; linear or non-linear.

*Linear inversion* techniques consists of dividing the subsurface into a series of geometric bodies. The anomalies created by this geometry are forward modelled and the mismatch between these anomalies and the observed measurements are used to change the initial model [32] with respect to the magnetic properties of the bodies. This problem can be formulated as a set of simultaneous equations, where the unknowns are the susceptibilities of each body. If the number of bodies equals the number of observations, an exact fit between observed and calculated fields will be obtained. If there are more observations than bodies, a least square solution will be required. The geometries of the bodies can have any arrangement, but using a matrix-type grid of the bodies is most common. Linear inversion techniques based on solving simultaneous linear equations have not found widespread application to the interpretation problem. A more applicable linear inversion technique has been formulated by Bott [11]. If the geometric bodies have identical dimensions and form a continuous layer at a constant depth, a set of convolution coefficients may be calculated. The magnetisation of the blocks will be determined by convolving the coefficients with the observed anomalies.

*Non-linear inversion* [51, 2] attempts to get a fit between the observed anomalies and calculated magnetic field by iteratively modifying the geometry of the model and/or the magnetisation of the bodies. These trial and error based algorithms vary the unknowns and calculate the resulting magnetic anomalies. Such algorithms calculates the magnetic field of the model and vary the unknown parameters in such a way that it converges towards a fit. The fit is commonly defined as the minimum of the sum of the squares of the difference between calculated and observed anomalies. The problem of convergence is not straightforward, since successive improvements in fit by progressive parameter variation, may not lead to the best fit. Non-linear inversion methods have proved useful in well-constrained situations and modelling may be much faster compared to interactive forward modelling. Unless the problem is well-constrained, a solution will still give a good fit between the calculated and observed field, but the resulting model will often be *geologically unrealistic* [32].

### 2.4 Current inversion methods

Parametric inversion methods determine the parameters of a few geometrically simple bodies by non-linear inversion. This is an overdetermined problem since a small
number of possible models are considered [8]. The methods require a great deal of prior knowledge, knowledge about the shape of the causative bodies, initial susceptibilities and their limits. A rapid method for 3D modelling of magnetic anomalies has been developed by Rao and Babu [63]. The observed magnetic anomaly contour map is assumed to be caused by a number of magnetic bodies. Each prismatic body is characterised by more than ten parameters, and all of them are considered to be unknowns. A solution is achieved by the use of a non-linear optimisation technique. The method has proven useful for the analysis of even complicated contour maps. Coles [16] introduced an additional facility which permitted a link between parameters; i.e., prisms with common faces may be adjusted simultaneously and magnetisation may be adjusted by identical amounts.

Inversion of potential field data by two-dimensional models has been presented using a generalised linear inversion approach [57]. The Lanczos [23] inverse is used to ensure a fast and safe convergence. The method includes analysing the estimation error. This method assumes good geological prior assumptions and is used to separate two layers, e.g., map the basement in poorly mapped areas. Problems of convergence using the generalised linear inverse can be solved effectively by the damped approximate inversion technique [61, 62]. The damping factors and relative threshold are optimally selected to retrieve the model parameters. This work also includes studies of the uniqueness of the resolved parameters.

Linear programming algorithms have proven useful in achieving a numerical solution to the inverse gravity problem [42, 17]. This approach takes into account the non-uniqueness of the solution. The subsurface is divided into a great number of rectangular prisms of unknown densities. The set of all solutions to this underdetermined problem is analysed by means of various convex diagrams of moments, giving the bounds of the parameters to be estimated [65].

Hjelt [38] used the iterative Gauss-Seidel method on linear interpretation of magnetic and gravimetric data. An efficient use of this method requires many assumptions, e.g., the bodies are placed so that the maximum influence of the body appears at the corresponding field point and the bodies should have a similar size with "narrow" anomalies. Such assumptions restrict the flexibility of the method.

Spector and Grant [68] developed the mathematical basis for the application of power spectrum analysis on contoured aeromagnetic maps. An ensemble of blocks at varying depth, width, thickness and magnetisation are considered as a statistical model. The shape of the power spectrum depends on the horizontal size, depth, thickness and depth extent of the blocks. Green [26] developed this method for the analysis of profile data. When the data are transformed to the frequency domain, average widths and depths can rapidly be calculated for a large number of anomalies caused by one or more of the magnetic bodies. Wang & Hansen [76] proposed an approach which assumes that the causative bodies are polyhedral and inverts for the position of these bodies by using the spectrum of the magnetic data. A method for automatic separation of two homogeneous layers may be accomplished by systematically estimating an average depth and magnetisation contrast for the
surface followed by an iterative method to adjust local depths using the Fourier transform [22]. A method for forward modelling and inversion of the power spectrum from magnetic anomaly data assuming that the crustal magnetic field is caused by vertical-sided and uniformly magnetised prisms is developed [21]. The solution of the inverse problem is achieved iteratively, starting from an initial set of model parameters.

Linear inversion using convolution filters or Fourier transform can be used to transform magnetic maps into susceptibility maps. Gunn [28] described the theory for designing such filters in the spectral domain. He also showed that if the relief on magnetic layers (with constant susceptibilities) are small, there is a linear relationship between the relief and the magnetic field. An accurate mapping of relief can be made using convolution filters [29]. The magnetisation of the modelled body has to be known to achieve a successful result, see figure 2.9 (from [29]).

A method for inversion of gravity data to detect basement relief constrained by the knowledge of depth at isolated points is proposed [47]. It consists essentially of a downward continuation of the observed anomaly, and the solutions are stabilised by introducing a damping parameter. A statistically significant estimate is obtained by using the interface depth at just a few isolated points.

A method for 3D linear inversion of gravity data to define the density distributions constrained by geologic information is proposed [7]. This technique is designed to replace tedious forward modelling procedures. The convergence of the solution is controlled by a Levenburg-Marquart damping parameter which may be varied at each iteration. The associated algorithm generates statistical measures of the solution quality. A 3D magnetic inversion method that determines a distribution of susceptibility using conjugate gradients is introduced by Pilkington [59]. The subsurface is divided into a 3D array of rectangular blocks, each with constant susceptibility. The inversion allows specifying smoothing and depth-weighting of the solution. Solving

**Figure 2.9:** Direct mapping of the relief on a magnetic interface using a convolution filter. The magnetic profile was input to a filtering routine and the depth profile was output.
the 3D magnetic inversion problem using a preconditioned conjugate algorithm has shown effective in keeping the computations to a manageable level.

Simulated annealing is a stochastic combinatorial optimisation technique which has proven capable of inverting magnetic data [18]. The subsurface is divided into smaller elemental blocks with uniform internal magnetisation. Simulated annealing was employed to calculate the magnetisation which caused a measured magnetic anomaly field curve.

A method for inversion problems was formulated by Backus and Gilbert in 1967 [5]. This method found the acceptable model “closest” to an initial estimate and was used iteratively until an acceptable fit was achieved. Green [27] demonstrated the use of a Backus-Gilbert approach on the inversion of gravity profiles.

Medeiros and Silva [52] have presented a method for simultaneous estimation of the total magnetisation direction and the three-dimensional spatial orientation based on magnetic measurements. The method is based on the moment inversion approach and is particularly suited to interpret compact, isolated or disjointed, but correlated magnetic sources.

To reduce the effect of interference between closely spaced anomalies, the concept of using the 3D enhanced analytic signal has been developed [41]. This high-resolution technique is used to image geological boundaries such as contacts and faults. This technique is particularly suitable when interference effects are considerable and/or when remanent magnetisations are not negligible.

2.5 Dealing with the non-uniqueness

The problem of inherent non-uniqueness of magnetic data, i.e., any measured magnetic anomaly can be reproduced by an infinitesimally thin zone of magnetic dipoles beneath the surface, this shows that from a mathematical point of view, there is no depth resolution inherent in the magnetic field data [49]. The fact that a magnetic survey contains a finite number of observations which includes noise, represents another source to the non-uniqueness. Many authors have discussed how to deal with these problems.

A paper by Al-Chalabi [3] discussed various factors contributing to the non-uniqueness in gravity and magnetic inverse problems: (i) the shapes of some bodies, e.g., cylinders, may result in more ambiguity than other shapes, (ii) incomplete knowledge of the full length of the anomaly, (iii) the simplification of the real problem to a 2D representation, (iv) sensor noise and (v) the regional background compared to local details.

L.B. Pedersen [58] presented a method for the constrained inversion of potential field data, tested on a gravimetric data set. The work starts pointing out the distinction between the model and the free parameters. It is a trade off between the level of detail in the model and the uncertainties in the estimation of the parameters;
a somewhat simpler model may result in a degraded anomaly fit, but improved parameter estimation. The reduction in number of free variables is conducted in two steps. First all available geological/geophysical knowledge is incorporated, which leads to models with parameters of large uncertainties. The next step is to fix or constrain some of the parameters by "intelligent" guesses, resulting in better determined parameters, but a less complicated model. We are now back to a trial and error based approach, but it is performed in a controlled way by the interpreter. The method also includes some analysis of the parameter uncertainties and enables the interpreter to reduce the effect of the ambiguity by encoding constraints to the inversion.

Another approach divided the subsurface into a large number of cells with fixed size and inverts the magnetic data with respect to the unknown susceptibilities. The non-uniqueness is solved by minimising some kind of objective function of the model. Green [27] minimises a weighted model norm with respect to the reference model, which allows the interpreter to guide the inversion. Last & Kubic [46] choose to minimise the total volume of the body which result in a compact and structurally simple model. These methods work well, but they are not flexible enough to handle detailed models with many layers.

Li & Oldenburg [49] introduced a new approach in 3D inversion of magnetic data. Their model was divided into a set of rectangular cells, each having a constant susceptibility. The number of cells was normally greater than the number of measurements, which gave an underdetermined problem. A multicomponent objective function is then formed which has the flexibility to generate different types of models. The objective function can correct for undesirable aspects: the concentration of susceptibilities near the surface, the excessive structure and the existence of negative susceptibilities. The objective function includes an optional reference model. An additional weighting function in the objective function may also be applied, enabling the interpreter to include qualitative or quantitative knowledge in the modelling.

A genetic algorithm that simultaneously generates a large number of different solutions to various potential field inverse problems is introduced by Boschetti et al [10]. In simple cases, a satisfactory description of the ambiguity domain inherent in potential field data can be obtained by analysing the ensemble of possible solutions. Information about the expected bounds on the unknown parameters as well as a measure of the reliability of the final solution can be calculated. The algorithm can be modified to address large dimensional problems. Currently the method is limited to reconstruct the contact between a single anomalous body and the background.

A generalised compact gravity inversion technique that achieves stable solutions by introducing a priori information about the main directions of mass concentration is presented [6]. This method is particularly applicable to constant linear density sources such as mineralisations along faults and intruded sills/dikes in a sedimentary basin. The maximum uncertainty of the correct source density must be known within 40 percent. The use of this method is not restricted to areas where the geological knowledge is well known, even weak a priori information may be sufficient to render
the method feasible.

If the gravity and magnetic effects are caused by common bodies, simultaneous inversion of the combined data set has the potential of providing a better solution and the ambiguity is reduced [39].

Inverse modelling methods have a number of problems when applied to magnetic anomaly measurements, arising from the inherent ambiguity in the forward model. Ambiguity in the forward model means that very different parameter values (e.g., very different subsurface distributions) can lead to very similar measurements. Turning this around, it follows that a good match between the measurements you actually took and the measurements predicted by a forward model, does not imply that the parameters in the model are close to the actual parameter values. In practice, inverse modelling is augmented with prior information, or regularisation, to give better results, but still, the basic and fundamental problem of ambiguity cannot be made to disappear.

The method developed and tested in this thesis can be considered a sophisticated method for inverse modelling, taking into account and blending in an optimal fashion, both the measurements and the prior information. The first step is to make a prior model structure, which includes the geometries of geological units and proposed parameters, i.e., susceptibilities at various locations or of various units. A forward model which maps the parameters into the observed measurements (if the model were correct) is then developed. Finally, inverse modelling is performed, i.e., finding a set of parameters which make the predicted measurements as close as possible to what is actually measured. These parameters can be taken as the estimate of the parameters, or inversion, based on the measurements.
Chapter 3

Minimum Mean-Square Error Estimation

In this chapter, the basis for the estimation method applied later to the inversion of aeromagnetic measurements, is explained. An estimate of the variables in vector \( x \) will be formed, given a measurement \( y \). In other words, find a function \( f \) such that \( \hat{x} \triangleq f(y) \) is an estimate of the random vector \( x \). In general, many criteria can be used to determine what the best estimator function, \( f \), is. For Gaussian random variables, with deterministic \( A \)-matrix, many statistical estimation methods will give similar results and in this work the mean-squared error criterion is used.

The following theory (based on [53] and [12]) starts with an introduction to Gaussian random variables and the statistical framework, followed by the explanation of the Minimum Mean-Square Error (MMSE) estimation method which later will be applied to estimating the magnetic properties of subsurface rocks. A brief discussion regarding various estimation methods is also included.

3.1 Gaussian random vectors

Gaussian vector density

Let \( x \) be a random (vector) variable with values in \( \mathbb{R}^n \), informally expressed as \( x \in \mathbb{R}^n \), and assume a Gaussian distribution. The function \( p_x \)

\[
p_x(u) = \frac{1}{\sqrt{(2\pi)^n|\Sigma_x|}} \exp \left[ -\frac{1}{2} (u - \bar{x})^T \Sigma_x^{-1} (u - \bar{x}) \right]
\]  

(3.1)

is said to be a multivariate (\( n \)-variate) Gaussian density function, i.e., \( x \sim \mathcal{N}(\bar{x}, \Sigma_x) \); \( det(\Sigma_x) > 0 \).
\( \bar{x} \in \mathbb{R}^n \) is the vector of mean or expected value of \( x \), \textit{i.e.,}

\[
\bar{x} = \mathbf{E}x = \int up_x(u)du
\]

and \( \Sigma_x \) is the covariance matrix of \( x \), \textit{i.e.,}

\[
\Sigma_x = \mathbf{E}(x - \bar{x})(x - \bar{x})^T = \mathbf{E}xx^T - (\mathbf{E}x)(\mathbf{E}x)^T = \int(u - \bar{x})(u - \bar{x})^T p_x(u)du
\]

**Confidence ellipsoids**

The vector \( \bar{x} \) and matrix \( \Sigma_x \) describe the uncertainty ellipsoids (or confidence ellipsoids) associated with the random vector \( x \):

\[
\{ v \mid (v - \bar{x})^T \Sigma_x^{-1}(v - \bar{x}) \leq \alpha \}
\]

gives a certain confidence level ellipsoid. The variable

\[
(v - \bar{x})^T \Sigma_x^{-1}(v - \bar{x})
\]

has a \( \chi^2 \) distribution with \( n \) degrees of freedom (\( n \) is the number of parameters to be estimated), so the exact correspondence can be found from the inverse Cumulative Distribution Function (CDF) of a \( \chi^2 \) variable. The centre of the ellipsoid is at the mean \( \bar{x} \). The semi-axes are given by the eigenvectors of \( \Sigma_x \), with each semi-axis length equal to the square root of the eigenvalue of \( \Sigma_x \).

As an example, suppose the eigenvalues of \( \Sigma_x \) are 10, 5 and 0.01:

\[
\Sigma_x = 10v_1v_1^T + 5v_2v_2^T + 0.01v_3v_3^T,
\]

where \( v_i \) are the orthonormal eigenvectors. Roughly speaking, the "uncertainty cloud" is flat: it extends around 2 or 3 units in two directions and much less, around 0.1, in a third direction. Several interpretations can be given:

- With high probability \( x \) lies near the 2-dimensional plane passing through \( \bar{x} \) and spanned by \( v_1 \) and \( v_2 \).
- one can accurately predict \( x \) in the direction \( v_3 \); in orthogonal directions \( x \) varies much more.
- The linear function of \( x \) given by \( v_3^Tx \) has low variance.

A simple and useful fact is that a linear (or affine) transformation of a Gaussian variable yields another Gaussian variable. If \( y = Ax + b \) then \( y \) has mean \( A\bar{x} + b \) and covariance \( A\Sigma A^T \).
3.1. GAUSSIAN RANDOM VECTORS

Figure 3.1: 100 samples of $x_1$ and $x_2$ plotted, with the 90% confidence ellipsoid; 91 out of 100 samples lies within the ellipsoid.

Checking the Gaussian assumption

To verify that the Gaussian assumptions for the prior parameters, e.g., $\bar{x}$, $\Sigma_x$, $\bar{v}$, and $\Sigma_v$ are reasonable, a simulation of these parameters can be performed. This is shown as an example [12] assuming that the following parameters for two variables of $x$: $x \sim N(\bar{x}, \Sigma_x)$

$$\bar{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\Sigma_x = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

- $x_1$ has mean 2 and std. dev. $\sqrt{2}$
- $x_2$ has mean 1 and std. dev. 1
- correlation coefficient between $x_1$ and $x_2$ is $\rho = \frac{1}{\sqrt{2}}$

Figure 3.1 shows 100 random samples according to the specified distribution. The 90% confidence ellipsoid is plotted and shows that in this case 91 samples are within the ellipsoid. The user should then accept that all these values are reasonable as prior information.
Jointly Gaussian vectors

Let \( x \) and \( y \) individually be \( n \)- and \( m \)-dimensional Gaussian random vectors, i.e., \( x \sim \mathcal{N}(\bar{x}, \Sigma_x) \) and \( y \sim \mathcal{N}(\bar{y}, \Sigma_y) \), \( x \in \mathbb{R}^n \) and \( y \in \mathbb{R}^m \).

Consider the Gaussian vector variable \([x^T \ y^T]^T \), \( x \in \mathbb{R}^n \), \( y \in \mathbb{R}^m \), with mean

\[
\mathbf{E}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix}
\]

and covariance

\[
\mathbf{E}\begin{bmatrix} x - \bar{x} \\ y - \bar{y} \end{bmatrix}\begin{bmatrix} x - \bar{x} \\ y - \bar{y} \end{bmatrix}^T = \Sigma = \begin{bmatrix} \Sigma_x & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_y \end{bmatrix}.
\]

Note that the diagonal blocks of \( \Sigma \) are the covariance matrices of the two variables \( x \) and \( y \), and the \((1, 2)\) entry is the (vector) covariance between them:

\[
\Sigma_{xy} = \mathbf{E}\{(x - \bar{x})(y - \bar{y})^T\}
\]

\[
\Sigma_{yx} = \mathbf{E}\{(y - \bar{y})(x - \bar{x})^T\} = \Sigma_{xy}^T
\]

Thus, \( x \) and \( y \) are uncorrelated if and only if \( \Sigma_{xy} = \Sigma_{yx} = 0 \).

Conditional density function

The conditional density function for \( x \) given \( y \) is given by:

\[
p(x|y) = \frac{p(x, y)}{p(y)} \tag{3.2}
\]

If \( x \) and \( y \) are \( n \)- and \( m \)-dimensional vectors that are jointly Gaussian, then (see [53, p167] for proof):

\[
p(x|y) = \frac{1}{\sqrt{(2\pi)^n\det(\Sigma_{x|y})}} \exp\left[-\frac{1}{2}(x - w)^T\Sigma_{x|y}^{-1}(x - w)\right], \tag{3.3}
\]

where:

\[
\Sigma_{x|y} = \Sigma_x - \Sigma_{xy}\Sigma_y^{-1}\Sigma_{xy}^T
\]

\[
w = \bar{x} + \Sigma_{xy}\Sigma_y^{-1}(y - \bar{y}) \tag{3.5}
\]

\( i.e., \) Gaussian with mean \( w \) and covariance \( \Sigma_{x|y} \).
3.2 Linear model with measurement noise

Consider the following equation:

\[ y = Ax + v \]  \hspace{1cm} (3.6)

where:

- \( y \in \mathbb{R}^m \) denotes the measurement vector, total magnetic field anomaly taken at various locations
- \( x \in \mathbb{R}^n \) denotes the random vector of variables, or parameters to be estimated
- \( v \in \mathbb{R}^m \) denotes the random vector of measurement noise
- \( A \in \mathbb{R}^{m \times n} \) is a known matrix that relates the variables to the measurements

The vector \( x \) represents in this context the physical parameter, \textit{susceptibility}, of bodies at various locations in the subsurface. The basic setup (3.6) can include 2D or 3D arrangements, even with multiple parameters. For example, \( x \) might represent the susceptibility on a 3D grid of pixels, \( 10 \times 15 \times 5 \). In this case, \( x \) is a vector with 750 components.

In this model, the vector, \( y \), contains \( m \) magnetic measurements. The important part is that the measurement \( y \) is a linear function of the variable \( x \). This will be explained in Chapter 4. This is a reasonable assumption when the geometries of the model is fixed; thus excluding cases where, for example, the components of \( x \) are geometric parameters such as strike length, depth, orientation and angle of the bodies, which enter the measurements \textit{nonlinearly}.

The vector \( v \) includes sensor bias (sensor noise) for \( m \) measurements, errors left after the time variation of magnetic anomalies is accounted for by subtracting a base station measurement.

3.3 Statistical model

The following statistical models for \( x \) and \( v \) are now used. The variables of vector \( x \) are jointly Gaussian, with mean vector \( \bar{x} \in \mathbb{R}^n \) and covariance matrix

\[ \mathbf{E}(x - \bar{x})(x - \bar{x})^T = \Sigma_x \in \mathbb{R}^{n \times n}. \]

Similarly, assume that the measurement errors in vector \( v \) are jointly Gaussian, with mean vector \( \bar{v} \in \mathbb{R}^m \) and covariance matrix

\[ \mathbf{E}(v - \bar{v})(v - \bar{v})^T = \Sigma_v \in \mathbb{R}^{m \times m}. \]

The assumptions are that \( x \) and \( v \) are independent, and that \( \bar{x}, \Sigma_x, \bar{v}, \) and \( \Sigma_v \) are known.
\( \hat{x} \) can be interpreted as the best prior guess of what \( x \) is, and \( \Sigma_x \) as an indication of how reliable this initial guess is. \( \bar{v} \) is interpreted as a constant offset or bias in the measurements. Since such bias is removed by some pre-processing before further signal processing, \( \bar{v} = 0 \), can be assumed. \( \Sigma_v \) is interpreted as a description of the measurement noise level, or reliability of the measurements. The assumption that the elements of \( \Sigma_v > 0 \) holds, i.e., there is always some (possibly small) non-zero measurement noise. This assumption simplifies the mathematics and is always reasonable in practice.

Some simple cases are now described. Assume that the components of \( x \) are independent, then \( \Sigma_x \) is a diagonal matrix, and \( \sqrt{\Sigma_{ii}} \) is the standard deviation of \( x_i \). Thus, for example, \( \hat{x}_i \pm 2\sqrt{\Sigma_{ii}} \) gives the 2\( \sigma \) confidence interval for \( x_i \). Conversely, suppose that the prior estimate \( \hat{x}_i \) is assumed accurate to \( \pm \sigma_i \) (standard deviation), then:

\[
\Sigma_x = \text{diag}(\sigma_1^2, \ldots, \sigma_n^2).
\]

This model can be used to specify a different reliability for each component \( x_i \); some can have very high reliability or confidence (i.e., small \( \sigma_i \)); others can have low reliability or confidence. The possibility that \( x_i \) is known perfectly is also allowed, which corresponds to \( \sigma_i = 0 \).

This model can be improved by adding correlations between the variables of \( x \), which corresponds to a non-diagonal matrix \( \Sigma_x \). In this way, the fact that it is reasonable that the value of a parameter \( x \) is somewhat similar to its neighbour parameter, can be encode. In the general case, the correlation between \( x_i \) and \( x_j \) is given by

\[
\rho_{ij} = \frac{\Sigma_{ij}}{\sqrt{\Sigma_{ii}\Sigma_{jj}}}.
\]  

(3.7)

where \( \Sigma_{ii} \) and \( \Sigma_{jj} \) are the variances of (the scalar random variable) \( x_i \) and \( x_j \), respectively and \( \Sigma_{ij} \) is the covariance between \( x_i \) and \( x_j \).

Note that

\[
\text{Tr} \Sigma = \sum_{i=1}^{n} \Sigma_{ii} = \mathbb{E}\|x - \hat{x}\|^2
\]

gives a natural scalar measure of the variance of the vector \( x \): the mean square deviation.

Suppose for example that \( x_i \) denotes the magnetic susceptibilities in a row of pixels (a matrix element representing a rectangular area of the subsurface) at some depth, and \( \hat{x}_i \) is the best prior estimate or guess of the susceptibility in pixel \( i \), \( \sigma_i \) is the reliability of this estimate, and adjacent pixel densities are correlated by the factor \( \rho \). This description corresponds to a (tridiagonal) matrix \( \Sigma \) with

\[
\Sigma = \begin{bmatrix}
\sigma_1^2 & \rho \sigma_1 \sigma_2 & 0 & \cdots & 0 \\
\rho \sigma_1 \sigma_2 & \sigma_2^2 & \rho \sigma_2 \sigma_3 & \cdots & 0 \\
0 & \rho \sigma_2 \sigma_3 & \sigma_3^2 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \sigma_n^2
\end{bmatrix}.
\]
Note that specifying a correlation between adjacent pixels allows incorporating the prior knowledge of how fast the parameter that \( x \) represents can change in space. In the 2D case, a correlation between adjacent horizontal pixels, and a different correlation between adjacent vertical pixels will be implemented.

All of these models of the prior information on \( x \) and the measurement errors \( v \) can be incorporated in the method. To summarize, the basic set-up requires the knowledge or specification of the following matrices and vectors:

- \( A \), the matrix relating variables to measurements
- \( \bar{x} \), the mean value vector of \( x \), i.e., our best prior guess of what the variables are
- \( \Sigma_x \), covariance matrix of \( x \), which describes the variability or reliability in the prior guess of vector \( \bar{x} \) for vector \( x \)
- \( \bar{v} \), vector the mean value (bias or offset) of the measurements
- \( \Sigma_v \), covariance matrix of \( v \), which describes the variability or repeatability of the measurements

Given these data, these two tasks have to be solved:

- **Estimation or inversion**: given a particular measurement \( y \), find \( \hat{x} \), the best posterior estimate of \( x \).
- **Reliability assessment**: determine the variability or confidence level in the posterior estimate \( \hat{x} \).

Many papers have been written addressing the first task. The second task, reliability assessment, is critical to the usefulness of magnetic anomaly inversion, since such measurements only give a fixed, and sometimes small, amount of additional information.

The Gaussian set-up seems to be a good approach. Prior to the estimation, the assumption of Gaussian variables should be evaluated.

### 3.4 Examples of correlations

First, a simple example is shown, in which \( x \in \mathbb{R}^4 \) represents the values of some physical quantity in a 2\( \times \)2 array of pixels, and \( y \in \mathbb{R}^3 \) represents three measurements taken above the array, as indicated in figure 3.2.

The matrix \( A \) that relates \( x \) to \( y \) is given by

\[
A = \begin{bmatrix}
1.0 & 0.3 & 0.5 & 0.2 \\
0.7 & 0.7 & 0.1 & 0.1 \\
0.3 & 1.0 & 0.2 & 0.5
\end{bmatrix}
\]
Sensors

\[
\begin{array}{ccc}
y_1 & y_2 & y_3 \\
\bullet & \bullet & \bullet \\
\end{array}
\]

\[
\begin{array}{c|c}
x_1 & \rho_h \\
\hline
\rho_v & x_2 \\
\hline
\rho_h & x_3 \\
\hline
\rho_v & x_4 \\
\end{array}
\]

pixels

**Figure 3.2:** Model with 4 pixels, their related correlations and 3 sensor measurements.

represents some kind of physical effect that decreases with the distance from pixel to sensor (*e.g.*, inverse square).

The statistical model is then described. The sensor noise is assumed zero-mean and independent with standard deviation 0.05. This is modelled by taking

\[
\bar{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \Sigma_v = \text{diag}(0.05^2, 0.05^2, 0.05^2).
\]

The statistical model of the pixel values is then developed, which encodes the prior knowledge about \( x \). Assume that \( x_1 \) and \( x_2 \), *i.e.*, the top level pixel values, have mean value 1, and standard deviation 0.5. The bottom level pixels, \( x_3 \) and \( x_4 \), have mean values 2 and standard deviations 0.8. Thus, the top level pixel values typically range between 0 and 3 (or so), and the bottom level pixels have typical values that range between 0.4 and 3.6. Finally, assume that the horizontal correlation, \( \rho_h \), between \( x_1 \) and \( x_2 \), and also \( x_3 \) and \( x_4 \), is 0.7, while the vertical correlation, \( \rho_v \), between \( x_1 \) and \( x_3 \), and also \( x_2 \) and \( x_4 \), is about 0.2. This might reflect the prior knowledge, *i.e.*, primarily horizontal layers, hence there is far more spatial correlation in the horizontal direction than in the vertical direction.

\[
\Sigma_{ij} = \rho_{ij} \cdot \sqrt{\Sigma_{ii} \Sigma_{jj}}
\]
All this information can be summarised as:

\[
\bar{x} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \quad \Sigma_x = \begin{bmatrix}
0.5^2 & 0.7 \cdot 0.5 \cdot 0.5 & 0.2 \cdot 0.5 \cdot 0.8 & 0 \\
0.7 \cdot 0.5 \cdot 0.5 & 0.5^2 & 0 & 0.2 \cdot 0.5 \cdot 0.8 \\
0.2 \cdot 0.5 \cdot 0.8 & 0 & 0.8^2 & 0.7 \cdot 0.8 \cdot 0.8 \\
0 & 0.2 \cdot 0.5 \cdot 0.8 & 0.7 \cdot 0.8 \cdot 0.8 & 0.8^2 \\
\end{bmatrix}
\]

The matrix \(\Sigma_x\) is intentionally left in a form that shows the origin of each of its entries. The zeros in the (1,4) and (2,3) entries (and also the (4,1) and (3,2) entries) reflects the model that \(x_1\) and \(x_4\), as well as \(x_2\) and \(x_3\), are uncorrelated.

**Modelling spatial correlation**

So far a simple model of the spatial correlation, such as a correlation between adjacent pixels, is described. This is meant only to show that the parameters do not tend to change very much over one pixel distance. It is possible to have a far more sophisticated model of the spatial covariance, that more accurately represents how the quantity is correlated between different, displaced points.

To explain the models, the assumption is made that \(x \in \mathbb{R}^n\) represents some quantity on a 2-dimensional grid of \(p \times q = n\) pixels. (It is straightforward to generalise to 3-dimensions.)

The vectors \(X \in \mathbb{R}^n\) and \(Z \in \mathbb{R}^n\) denote the \(X\) and \(Z\) positions of the (centre of the) pixels. Thus, \(x_i\) is the value of the physical variable in a pixel centred at position \((X_i, Z_i)\), and the distance between pixel \(i\) and pixel \(j\) is given by

\[
d_{ij} = \sqrt{(X_i - X_j)^2 + (Z_i - Z_j)^2}
\]

The simplest spatial correlation (or random field) model assumes that each \(x_i\) has the same standard deviation and that the correlation between \(x_i\) and \(x_j\) is a function of the distance between them, for example:

\[
\Sigma_{ij} = a^2 e^{-(X_i - X_j)^2 + (Z_i - Z_j)^2}/D^2,
\]

where \(a\) is the standard deviation of the \(x_i\)'s, and \(D\) is called the decorrelation distance. In this model, the correlation between pixels drops off exponentially with increasing distance. When two pixels are separated by a distance \(D\) apart, they are correlated by \(e^{-1}\), about 37%.

A more sophisticated version can capture the idea that the correlations in space can be different in the horizontal and vertical directions:

\[
\Sigma_{ij} = a^2 e^{-(X_i - X_j)^2 + (Z_i - Z_j)^2}/R^2
\]

where \(R = R^T \in \mathbb{R}^{2 \times 2}\) is a matrix that characterises the spatial variations. By taking \(R = \text{diag}(d_X^2, d_Z^2)\), a correlation distance \(d_X\) is the \(X\)-direction and a correlation
distance $d_Z$ in the $Z$-direction is specified. More generally, the ellipsoid determined by $R$, i.e.,

$$\{ \mathbf{u} \in \mathbb{R}^2 \mid \mathbf{u}^T R^{-1} \mathbf{u} \leq 1 \},$$

where

$$\mathbf{u} = \begin{bmatrix} X_i - X_j \\ Z_i - Z_j \end{bmatrix}$$

shows the *decorrelation ellipsoid*, i.e., a picture of the region which is correlated at least by $e^{-1}(\sim 0.37)$.

Figure 3.3 shows two examples of distance correlation matrices.

### 3.5 Estimation and regularisation methods

In general, many criteria can be used to determine what the best estimator function $f$ is. In the following some alternative estimation methods are commented [53]:

3.5. ESTIMATION AND REGULARISATION METHODS

- Minimum Mean-Square Error Estimation
- Maximum-likelihood Estimation
- Maximum a Posteriori Estimation
- State Estimation (Kalman Filter)
- Regularisation methods

Suppose \( x \in \mathbb{R}^n \) and \( y \in \mathbb{R}^m \) are random vectors (not necessarily Gaussian). Calculate an estimate of \( x \), denoted \( \hat{x} \), such that:

\[
\hat{x} = f(y)
\]

is near \( x \); \( f \) is a function from \( \mathbb{R}^m \rightarrow \mathbb{R}^n \)

The MMSE estimation method determines a function \( f \), such that the mean square prediction error, \( E \|f(y) - x\|^2 \), is minimised. The general solution is:

\[
f(y) = E(x|y),
\]

i.e., the conditional expectation of \( x \) given \( y \).

The Maximum-Likelihood (ML) method is based on the idea that different populations generate different samples, and that any given sample is more likely to come from some populations than from others. Because likelihood is proportional to probability, the entire joint probability density function has to be known in order to determine the formula for the likelihood function. If \( A \) is deterministic and \( x \) Gaussian, then the ML estimator and the MMSE estimator is the same.

Maximum a Posteriori (MAP) estimation of parameters is also known as Bayesian estimation. Obtaining a MAP estimate involves specifying both \( p(y|x) \) and \( p(x) \) and finding the value of \( x \) that maximises \( p(x|y) \). When \( y \) and \( x \) are jointly Gaussian, then the MAP estimator and the MMSE is the same.

The Kalman filter is a way to efficiently implement the MMSE when the measurements arrive periodically, and what you are estimating is changing according to some dynamics. In this work, KF is not really relevant because what is estimated in this work is not changing with time. The static version of the Kalman Filter is a way of implementing MMSE estimation.

Finally it is mentioned that the estimator formula is a kind of regularisation, i.e., a way of making a guess at an inversion when the problem is underdetermined:

\[
\min \| Ax - y \| 
\]

where \( A \in \mathbb{R}^{m \times n} \) is an underdetermined matrix, \( n \geq m \). In general a regularisation solution \( x_\lambda \) is defined as the unique solution of the following least-squares problem:

\[
\min (\| Ax - y \|^2 + \lambda \| Px \|^2) 
\]
where $\lambda$ is an regularisation parameter that controls the “smoothness” of the regularised solution, and $P$ is a regularisation operator, which may be determined in an “ad hoc” manner. Selecting $P = I$, equation 3.10 is transformed to the highly used regularisation method due to Tikonov. Hansen [36] has introduced the use of truncated singular value decomposition (TSVD) to solve the discrete ill-posed problem. The Backus-Gilbert [5] was designed as an alternative to the classical inversion methods, and Hansen [37] has shown the use of SVD to analyse the linear systems of equations.

The difference between the MMSE estimate and standard regularisation techniques is that MMSE uses a statistical basis for setting the regularisation parameters.

In this thesis, a statistical model is used, i.e., the variables are associated with a statistical distribution (Gaussian) and the noise is Gaussian. The $A$-matrix, which relates the variables to measurements, is deterministic. Under these conditions, the various statistical methods discussed above, turn out to be the same. The MMSE estimation method which minimises the mean square prediction error, $\mathbb{E} \| f(y) - x \|^2$ is chosen.

### 3.6 MMSE estimation

The MMSE estimation method determines a function $f$, such that the mean square prediction error, $\mathbb{E} \| f(y) - x \|^2$, is minimised. In the general case the optimal function $f$ for least-mean-square prediction is the conditional mean, i.e., (proof in [53, p176]):

$$f(y) = \mathbb{E}(x|y) \quad (3.11)$$

When $x$ and $y$ are jointly Gaussian, using equations 3.11 and 3.5, the estimator (MMSE) that minimises the mean-squared error is:

$$\hat{x} = \bar{x} + \Sigma_{xy} \Sigma_y^{-1}(y - \bar{y}) \quad (3.12)$$

Since $x$ and $v$ are assumed Gaussian and mutually uncorrelated, they are jointly Gaussian; consequently $y$ and $x$ are also jointly Gaussian, thus $\hat{x}$ is given by equation 3.12. Consider the generic linear model:

$$y = Ax + v \quad (3.13)$$

This leads to the following: for the generic linear model in which $A$ is deterministic and $v$ is white Gaussian noise with known covariance matrix $\Sigma_v$,

$$x \sim N(\bar{x}, \Sigma_x) \quad (3.14)$$

and $x$ and $v$ are mutually uncorrelated, then the MMSE estimate of $x$ is:

$$\hat{x} = \bar{x} + \Sigma_x A^T(A \Sigma_x A^T + \Sigma_v)^{-1}(y - A\bar{x}) \quad (3.15)$$
3.6. MMSE ESTIMATION

Proof: Using equations 3.13 and 3.14 it can be shown that:

\[ \tilde{y} = A\hat{x} \]  \hspace{1cm} (3.16)

using the fact that \( x \) and \( v \) are mutually uncorrelated:

\[ \Sigma_y = \mathbb{E}\{(y - \bar{y})(y - \bar{y})^T\} \]
\[ = \mathbb{E}\{(Ax + v - A\bar{x})(Ax + v - A\bar{x})^T\} \]
\[ = \mathbb{E}\{A(x - \bar{x}) + v)(A(x - \bar{x}) + v)^T\} \]
\[ = \mathbb{E}\{A(x - \bar{x})(x - \bar{x})^T\}A^T + \mathbb{E}\{vv^T\} \]
\[ = A \mathbb{E}\{(x - \bar{x})(x - \bar{x})^T\}A^T + \Sigma_v \]
\[ \Sigma_y = A\Sigma_x A^T + \Sigma_v \]  \hspace{1cm} (3.17)

\[ \Sigma_{xy} = \mathbb{E}\{(x - \bar{x})(y - \bar{y})^T\} \]
\[ = \mathbb{E}\{(x - \bar{x})(Ax + v - A\bar{x})^T\} \]
\[ = \mathbb{E}\{(x - \bar{x})(A(x - \bar{x}) + v)^T\} \]
\[ = \mathbb{E}\{(x - \bar{x})(x - \bar{x})^T\}A^T \]
\[ \Sigma_{xy} = \Sigma_x A^T \]  \hspace{1cm} (3.18)

By substituting equations 3.16 - 3.18 into equation 3.12, equation 3.15 is verified.

Denote the error-covariance matrix associated with \( \hat{x} \) for \( \Sigma_{est} \), then:

\[ \Sigma_{est} = \Sigma_{(x|y)} = \Sigma_x - \Sigma_x A^T(A\Sigma_x A^T + \Sigma_v)^{-1} A\Sigma_x \]  \hspace{1cm} (3.19)

Proof: The estimation error, \( \hat{x} \), is defined by:

\[ \hat{x} = (x - \bar{x}) \]  \hspace{1cm} (3.20)

Since \( \hat{x} \) is unbiased, \( i.e., \mathbb{E}(\hat{x}) = 0 \), then:

\[ \Sigma_{est} = \mathbb{E}(\hat{x}\hat{x}^T) \]  \hspace{1cm} (3.21)

From equations 3.15 and 3.20:

\[ \hat{x} = (x - \bar{x}) - \Sigma_x A^T(A\Sigma_x A^T + \Sigma_v)^{-1}(y - A\bar{x}) \]  \hspace{1cm} (3.22)

From equations 3.22 and 3.21 the following expression is derived:

\[ \Sigma_{est} = \mathbb{E}\{(x - \bar{x})(x - \bar{x})^T\} - \mathbb{E}\{(x - \bar{x})[\Sigma_x A^T\Sigma_y^{-1}(y - A\bar{x})][y^T]\} \]
\[ - \mathbb{E}\{[\Sigma_x A^T\Sigma_y^{-1}(y - A\bar{x})][\Sigma_x A^T\Sigma_y^{-1}(y - A\bar{x})][x - \bar{x}]^T\} + \mathbb{E}\{[\Sigma_x A^T\Sigma_y^{-1}(y - A\bar{x})][\Sigma_x A^T\Sigma_y^{-1}(y - A\bar{x})][x - \bar{x}]^T\} \]

The terms in equation 3.23 are calculated separately: the first term is the definition of the covariance matrix

\[ \mathbb{E}\{(x - \bar{x})(x - \bar{x})^T\} = \Sigma_x \]  \hspace{1cm} (3.23)
Calculating the second term, using equations 3.13 and 3.23, and the fact that \( v \) and \( x \) are independent:

\[
\begin{align*}
- \mathbf{E}\{ (x - \bar{x})(\Sigma_x A^T \Sigma_y^{-1}(y - A\bar{x}))^T \} \\
= - \mathbf{E}\{ (x - \bar{x})(\Sigma_x A^T \Sigma_y^{-1}(Ax + v - A\bar{x}))^T \} \\
= - \mathbf{E}\{ (x - \bar{x})(\Sigma_x A^T \Sigma_y^{-1}(Ax - A\bar{x}))^T + \mathbf{E}\{ (x - \bar{x})(\Sigma_x A^T \Sigma_y^{-1}v)^T \} \\
= - \mathbf{E}\{ (x - \bar{x})(\Sigma_x A^T \Sigma_y^{-1}A(x - \bar{x}))^T \} \\
= -\Sigma_x A^T \Sigma_y^{-1}A\Sigma_x,
\end{align*}
\]

The third term is calculated like this:

\[
\begin{align*}
- \mathbf{E}\{ (\Sigma_x A^T \Sigma_y^{-1}(y - A\bar{x}))^T(x - \bar{x}) \} \\
= - \mathbf{E}\{ \Sigma_x A^T \Sigma_y^{-1}A(x - \bar{x})(x - \bar{x})^T \} + \mathbf{E}\{ \Sigma_x A^T \Sigma_y^{-1}v(x - \bar{x}) \} \\
= -\Sigma_x A^T \Sigma_y^{-1}A\Sigma_x
\end{align*}
\]

And the last term:

\[
\begin{align*}
\mathbf{E}\{ (\Sigma_x A^T \Sigma_y^{-1}(y - A\bar{x}))(\Sigma_x A^T \Sigma_y^{-1}(y - A\bar{x}))^T \} \\
= \mathbf{E}\{ (\Sigma_x A^T \Sigma_y^{-1}(y - \bar{y}))(\Sigma_x A^T \Sigma_y^{-1}(y - \bar{y}))^T \} \\
= \mathbf{E}\{ \Sigma_x A^T \Sigma_y^{-1}(y - \bar{y})(y - \bar{y})^T(\Sigma_x A^T \Sigma_y^{-1})^T \} \\
= \Sigma_x A^T \Sigma_y^{-1}A\Sigma_x
\end{align*}
\]

Combining the four terms, and using equation 3.17, equation 3.19 is verified.

Thus, for Gaussian variables the optimal \( f \) has a simple form:

\[
\hat{x} = \mathbf{E}(x|y) = \bar{x} + B(y - \bar{y}), \tag{3.24}
\]

where

\[
B = \Sigma_x A^T (A\Sigma_x A^T + \Sigma_v)^{-1}. \tag{3.25}
\]

where \( \bar{y} = A\bar{x} + \bar{v} \) is the expected value of the sensor measurement \( y \) and \( B \) is derived from equation 3.15. (Since \( \Sigma_v > 0 \) is assumed, the inverse in this formula always exists.)

This important formula can be interpreted as follows: \( \bar{x} \) is the mean of \( x \), which is the minimum mean-square estimate of \( x \) given no information; the other term is a linear correction based on \( y - \bar{y} \), which is the difference between the actual sensor measurement \( y \) and the expected sensor measurement \( \bar{y} \).

Note that \( A\Sigma_x A^T \) can be interpreted as the "signal covariance" (in the measurement) and \( \Sigma_v \) is the noise covariance. So, roughly speaking, the estimator matrix \( B \) depends on the "signal to noise ratio" at the sensor. \( B \) is large when the signal-to-noise ratio is high, and small when it is low.

The estimation error, which is the random variable \( \hat{x} = (x - \hat{x}) \), is also a Gaussian variable with mean zero and covariance \( \Sigma_{est} \), gives the reliability measure needed; it
3.6. MMSE ESTIMATION

describes how close the estimate \( \tilde{x} \) will be to \( x \), or how reliable the “inversion” or “interpretation” \( \tilde{x} \) is. For example, the standard deviation (\( \sigma_x \)) of the estimate of \( x_i \) is given by \( \sqrt{\Sigma_{\text{est} ii}} \) and the “2\( \sigma_x \)” confidence interval for \( x_i \) is given by \( \tilde{x}_i \pm 2\sqrt{\Sigma_{\text{est} ii}} \).

The formula (3.19) is instructive; based on the following assumptions:

- \( A > 0 \),
- \( \Sigma_x > 0 \), i.e., all \( x \)'s are \textit{not known} precisely,
- \( \Sigma_x > 0 \), i.e., there is always some sensor noise,

the conclusion is that:

\[
\Sigma_{\text{est}} < \Sigma_x,
\]

which means that the posterior variance is always less than the prior variance; in other words, the measurements \textit{always improves} the estimate of \( x \). Indeed the difference, \( \Sigma_x - \Sigma_{\text{est}} \), serves as a measure of the value of the measurements, or the information obtained by the measurements.

More specifically, focus on \( x_i \). The prior “\( \sigma \)” confidence interval is

\[
\tilde{x}_i \pm \sqrt{\Sigma_{x ii}}.
\]

After the measurement \( y \) is taken, the calculated \textit{posterior} or refined interval is

\[
\tilde{x}_i \pm \sqrt{\Sigma_{\text{est} ii}}.
\]

The posterior interval is always smaller than the prior interval. The ratio of the posterior to prior uncertainty interval widths, i.e.,

\[
\frac{\sqrt{\Sigma_{\text{est} ii}}}{\sqrt{\Sigma_{x ii}}}
\]

gives the amount of uncertainty reduction in \( x_i \) that is provided by the measurement \( y \). This number is always between 0 and 1. If it is nearly one, then the measurement \( y \) provides very little new information about \( x_i \). This is a disappointing, but at least honest outcome: it shows that the aeromagnetic measurements, simply doesn’t contain much information about \( x_i \). On the other hand if the ratio is small, the measurement is able to greatly reduce the uncertainty in \( x_i \); in particular attach a great confidence to their estimate \( \tilde{x}_i \).

Another observation is the possibility of calculating the variance of the estimate, \( \Sigma_{\text{est}} \) \textit{before} actually making any measurements. Thus the information value provided by a set of measurements can be assessed \textit{before} actually performing them. Indeed the only information needed is to know \( A \), which characterises the sensors, \( \Sigma_x \), and \( \Sigma_\nu \) to calculate \( \Sigma_{\text{est}} \). This raises the possibility of experiment design, i.e., deciding what measurements to take, before taking any data, see section 5.10. To find the estimate \( \hat{x} \), \( \bar{x} \), \( \bar{v} \), and of course the measurement \( y \) are needed.
3.7 Quantifying model estimation

$\Sigma_x$ determines the prior knowledge about the vector $x$; $\Sigma_{\text{est}}$ determines the estimation error after the measurement is made. These matrices determine the shape and size of the prior and posterior confidence ellipsoids. It is already known that the posterior ellipsoid is "smaller" in all directions than the prior ellipsoid (i.e., $\Sigma_{\text{est}} < \Sigma_x$). Two different measures will be used to judge how much smaller $\Sigma_{\text{est}}$ is than $\Sigma_x$.

The first measure looks directly at the ability to estimate the $i$th parameter $x_i$. The prior (one $\sigma$) uncertainty interval has width $\sqrt{\Sigma_{ii}}$, whereas the posterior (one $\sigma$) uncertainty interval has width $\sqrt{\Sigma_{\text{est} \, ii}}$. The ratio,

$$\lambda_i = \frac{\sqrt{\Sigma_{\text{est} \, ii}}}{\sqrt{\Sigma_{ii}}}$$  \hspace{1cm} (3.26)

which is a scalar and is always $< 1$, explains how much better the estimate $x_i$ is after the measurement is taken. If these numbers are near one, the measurement does not improve the knowledge; if they are significantly smaller than one, the measurement substantially increases the knowledge. Note that there is one $\lambda$ for every element in $x$.

The second measure of estimation improvement is based on RMS (root mean square) variability. The RMS value of $x$ is given by

$$\sqrt{\mathbb{E} \frac{x_1^2 + \cdots + x_n^2}{n}} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} \Sigma_{ii}} = \sqrt{\text{Tr}(\Sigma_x)/n}$$  \hspace{1cm} (3.27)

where Tr denotes the trace of a matrix, i.e., the sum of its diagonal elements. This number gives the root mean square deviation of $x$ from its mean value. Now, after the measurement is made, the RMS value of the estimation error is given by

$$\sqrt{\frac{1}{n} \sum_{i=1}^{n} \Sigma_{\text{est} \, ii}} = \sqrt{\text{Tr}(\Sigma_{\text{est}})/n}.$$  \hspace{1cm} (3.28)

The ratio of posterior to prior RMS deviation,

$$\eta = \sqrt{\frac{\text{Tr}(\Sigma_{\text{est}})}{\text{Tr}(\Sigma_x)},}$$  \hspace{1cm} (3.29)

is smaller than one, and gives a scalar measure of the overall improvement in reliability based on the estimation, whereas $\lambda_i$ gives a similar measure for the component $x_i$. If $\eta$ is near one, then the measurement does not much improve the RMS error in predicting $x$; if it is substantially less than one, the measurement has significantly improved the RMS error of the estimate.

The forward fit between the anomalies based on the estimated parameters $y_{\text{est}}$ and the measured anomalies $y$, $r(\tau)$ is calculated by:
\[ \tau = \frac{||y - y_{est}||}{||y||} \] (3.30)

In summary, three measures will be calculated, reporting how informative a measurement is in the estimation. The numbers \( \lambda_1, \ldots, \lambda_n \), which quantifies the improvement in reliabilities of the estimate \( x_i \) after the measurement, compared to the prior reliabilities. The single number \( \eta \), which compares the RMS variability of \( x \) and the posterior estimation error. Finally the number \( \tau \), which compares the RMS error between forward fit of the estimated model with the measured anomalies, measuring the misfit in %.

In addition, the change in values of the components in \( x \), will be calculated to study the effect of the estimation process. The absolute value of the difference between the prior and the estimated values,

\[ \xi_i = |\bar{x}_i - \hat{x}_i| \] (3.31)

give information of the change in value for each pixel (region) due to the estimation, and

\[ |\ddot{x}| = |x - \hat{x}| \] (3.32)

denotes the absolute estimation error (when the true model is known).
Chapter 4

Forward Model for Magnetic Anomalies

4.1 Coordinates and pixels

Points are represented by their rectangular coordinates \((x, y, z)\). \(z\) denotes the depth (in meters) below some reference level (usually sea level), with positive \(z\) increasing downwards. \(x\) represents a distance from a reference point, in meters, in some fixed horizontal direction, usually along the track of interest. \(y\) represents the distance from the reference point along a horizontal direction orthogonal to the \(x\) direction. The three unit vectors form a right-handed system. To completely fix the coordinate system with respect to the earth, the latitude, longitude and elevation of the reference point \((0, 0, 0)\), as well as the direction of the \(x\)- or \(y\)- unit vector (i.e., track direction or cross track direction) need to be determined.

This study is only concerned with two dimensional scenarios, in which it is assumed that the susceptibility is homogeneous and extends infinitely in the \(\pm y\)-directions. Therefore points are given by their \((x, z)\) coordinates, the \(y\) direction is parallel to the strike, and the \(x\) direction is perpendicular to the strike.

A rectangle is divided in the \((x, z)\)-plane into \(n_x n_z\) pixels, denoted

\[
P_{ij}, \quad i = 1, \ldots, n_z, \quad j = 1, \ldots, n_x,
\]

where \(n_x\) is the number of pixels across the rectangle and \(n_z\) the number of pixels down the rectangle.

The lengths \(\Delta X\) and \(\Delta Z\) denote the \(x\) and \(z\) dimensions, respectively, of the pixels. Pixel \(P_{ij}\) can be described in coordinates as the rectangle

\[
P_{ij} = \{ (x, z) \mid (j - 1)\Delta X \leq x \leq j\Delta X, \ (i - 1)\Delta Z \leq z \leq i\Delta Z \}
\]

Thus, the index \(i\) gives the depth of the pixel, and \(j\) gives the \(x\) coordinate. More precisely, the pixel \(P_{00}\) has its top left corner at the reference point \((0, 0)\), and extends
right to \( x = \Delta X \) and down to \( z = \Delta Z \); more generally, pixel \( P_{ij} \) has its top left corner at the point \((j\Delta X, i\Delta Z)\), and extends right to \( x = (j + 1)\Delta X \) and down to \( z = (i + 1)\Delta Z \). The centre of pixel \( P_{ij} \) is given by

\[
\text{center } P_{ij} = ((j - 1/2)\Delta X, (i - 1/2)\Delta Z).
\]

The overall rectangle of \( n_xn_z \) pixels extends to the right \( n_x\Delta X \), and down \( n_z\Delta Z \).

A real-valued parameter or attribute \( x \) of a pixel, such as susceptibility or density, will be represented as an \( n_x \times n_z \) matrix \( x \), with \( x_{ij} \) denoting the (assumed constant over the pixel) value of the parameter or attribute \( x \) in pixel \( P_{ij} \).

### 4.2 Assumptions

The following terms are defined (see [25, ch11]):

- ambient field (main magnetic field of the earth), \( H_0 \)
- total magnetic field, \( H \)
- variations in magnetic total field, \( \Delta H \)
- measurements: changes in total magnetic field, \( \Delta T \)

The following assumptions are applied:

- negligible permanent magnetic moment (remanence)
- small variations in total field, \( i.e., |\Delta H| \ll H_0 \)
- the variations in total field is in direction of the ambient field

The causative bodies to be modelled is of course 3D in nature, but a 2D modelling algorithm is applied in this study. The simple 2D modelling assumes infinite length of the bodies perpendicular to the measurement direction. The length of the bodies on each side of the measurements is denoted, \( \Delta Y \), and the width of the bodies (along the measurement direction), \( l \). A practical approach for applying 2D methods in a 3D situation, is to assume:

- the strike of the bodies is perpendicular to the measurement direction
- \( \Delta Y > l \)

The sensor noise model is simple: \( v \) is zero mean, and \( \Sigma_v = \sigma^2_v I \), \( i.e., v_i \) are independent, zero mean, with standard deviation \( \sigma_v \) (white noise). More sophisticated models could include correlation between nearby measurements, but is not applied here.

Two different algorithms are implemented and tested out, a rectangular model and a point model for pixels.
4.3 Rectangular model

The two-dimensional case is now considered. This model is adapted from [45, p440]. The total field anomaly measurement at location \((x_s, z_s)\) is considered, due to a pixel (rectangle) with top left at \((x, z)\), top right at \((x+\Delta X, z)\), bottom left at \((x, z+\Delta Z)\), and bottom right at \((x+\Delta X, z+\Delta Z)\), with uniform, unit dipole moment per unit volume, see figure 4.1.

\[
\Delta T_a = p \log \frac{r_1 r_3}{r_2 r_4} + q(\theta_1 - \theta_2 + \theta_3 - \theta_4)
\]  

(4.1)

where

\[
\begin{align*}
r_1 &= \|(x_s, z_s) - (x, z)\|, \\
r_2 &= \|(x_s, z_s) - (x + \Delta X, z)\|, \\
r_3 &= \|(x_s, z_s) - (x + \Delta X, z + \Delta Z)\|, \\
r_4 &= \|(x_s, z_s) - (x, z + \Delta Z)\|
\end{align*}
\]

are the distances from the measurement location to the the pixel vertices,

\[
\begin{align*}
\theta_1 &= \angle((x, z) - (x_s, z_s)), \\
\theta_2 &= \angle(x + \Delta X, z) - (x_s, z_s)), \\
\theta_3 &= \angle(x + \Delta X, z + \Delta Z) - (x_s, z_s)), \\
\theta_4 &= \angle(x, z + \Delta Z) - (x_s, z_s))
\end{align*}
\]

are the angles of the vectors from the sensor location to the pixel vertices, and \(p\) and \(q\) are two parameters that depend on the orientation of the coordinate system with respect to the ambient magnetic field and are described below.

\[
p = -\cos \alpha \sin(2\beta), \quad q = \sin^2 \beta - \cos^2 \alpha \cos^2 \beta
\]

where \(\alpha\) denotes the azimuth of the ambient earth magnetic field, measured from the positive \(x\)-axis, and \(\beta\) denotes the dip or inclination of the ambient earth magnetic field. Thus, the ambient field is given by

\[
H_0(\cos \alpha \cos \beta, \sin \alpha \cos \beta, \sin \beta)
\]

where \(H_0\) is the magnitude of the ambient field.

The point model for pixels [25, p310] is also implemented and tested. In this model, all the dipole moment contained in a pixel is concentrated at the centre of the pixel. This model assumes the ambient field is in the \((x, z)\) plane, i.e., \(\alpha = 0\). Compared to the rectangular model this means

\[
p = -\sin 2\beta, \quad q = -\cos 2\beta
\]

The model is:

\[
\Delta T_b = \frac{\Delta X \Delta Z (2 \cos^2 \theta - 1)}{\|(x_c, z_c) - (x_s, z_s)\|^2}
\]
Figure 4.1: Geometries and parameters for the rectangular model.

<table>
<thead>
<tr>
<th>Model</th>
<th>$x$</th>
<th>$\Delta X$</th>
<th>$z$</th>
<th>$\Delta Z$</th>
<th>$\Delta T_a$</th>
<th>$\Delta T_b$</th>
<th>$\frac{\Delta T_a}{\Delta T_b}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>M1</td>
<td>300</td>
<td>200</td>
<td>200</td>
<td>400</td>
<td>0.2585</td>
<td>0.2165</td>
<td>1.1940</td>
</tr>
<tr>
<td>M2</td>
<td>300</td>
<td>200</td>
<td>600</td>
<td>400</td>
<td>0.0567</td>
<td>0.0541</td>
<td>1.0477</td>
</tr>
<tr>
<td>M3</td>
<td>600</td>
<td>200</td>
<td>1000</td>
<td>400</td>
<td>0.0266</td>
<td>0.0261</td>
<td>1.0177</td>
</tr>
<tr>
<td>M4</td>
<td>600</td>
<td>100</td>
<td>1000</td>
<td>100</td>
<td>0.0043</td>
<td>0.0043</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

Table 4.1: Magnetic anomalies for four separate pixels calculated with two forward methods: the rectangular model and the point model for pixels; $x$, $\Delta X$, $z$ and $\Delta Z$ in meters, $\Delta T_a$ and $\Delta T_b$ in nT.

where $x_c = x + \Delta X/2$, $z_c = z + \Delta Z/2$ are the coordinates of the centre of the pixel, and $\theta$ is the angle between the vector from the sensor location to the centre of the pixel, and the direction of the ambient earth field. Thus

$$\theta = \arctan(x_c - x_s, z_c - z_s) - \alpha.$$  

The point model is less accurate compared to the rectangular model, the deviation is greater when the measurement is close to the pixel and when the size of the pixel is large. Figure 4.2, shows four pixels with the calculated anomalies listed in table 4.1.

The selected rectangular model has been tested with several typical geological structures, such as different layer configurations and faults with various angles and lengths. A commercial software package [55] has also been used and the results show a good match.
Figure 4.2: Geometries of four rectangles used in the testing of two forward modelling methods; the measurement is located at point (0,0).
To enable effective and fast testing of the method presented in this thesis, the implementation is limited to the 2D case. The vector $x$ represents the pixel values for the 2D image of one vertical cross section of the subsurface (see figure 4.3 a).

A two-dimensional matrix, $A$, is calculated based on the geometry of the pixels and relates the variables $(x)$ to the measurements $(y)$. For each of the $m$ measurement points, one column of $A$ is calculated using equation 4.1 for each of the rectangles composing the model. The entries in $A$ is multiplied by the magnitude of the ambient field $H_0$. Each of the $m$ columns in $A$ contains a $n$ vector mapping the $n$ rectangles in the model $(x)$ to the $m$ measurements. The magnetic anomaly at a location above the surface is related to the subsurface susceptibilities by a linear relationship [49], consequently the forward model is linear. Figure 4.3 b demonstrates how the measurement vector $y$ is calculated based on vector $x$ and the columns in the $A$-matrix.

4.4 Region and gradient models

Regions with constant parameters

Suppose that the pixels have been grouped into $r$ given subsets which is refer to as regions. The regions are labelled as $R_1, \ldots, R_r$.

In the simplest model, the value of $x$ is assumed constant in each region. Let $z_i$ denote the value in $R_i$, so $z \in \mathbb{R}^r$. Introduce the matrix $F$ that maps the region values into the pixel values:

$$x = Fz.$$  

The matrix $F$ has the simple form

$$F_{ij} = \begin{cases} 1 & \text{pixel } i \text{ is in region } j \\ 0 & \text{otherwise.} \end{cases}$$  

The problem is then to estimate $z$ and analyse as usual using $AF$ in place of $A$. The forward modelled anomaly is calculated according to:

$$y = AFz + v$$

Regions with gradients in parameters

A more sophisticated model includes horizontal and vertical gradients in each region. Assume that the value of $x$ in each region is affine, i.e., linear plus constant:

$$x_i = z_k + g_k(X_i - \bar{X}_k) + h_k(Z_i - \bar{Z}_k)$$  

where:
Figure 4.3: Model parameters: a) shows the pixel matrix representing the subsurface geometry (with \( n \) pixel elements). The vector \( y \) contains \( m \) measurements. Height specifies the distance between the sensor and the uppermost layer. b) To calculate a measurement \( y \) at position \( i \), the \( x \)-vector is multiplied with column vector \( i \) in the \( A \)-matrix; sensor noise, \( v \), is assumed zero.
• $x_i$ is the value at pixel $i$

• Pixel $i$ is in region $k$

• $z_k$ is the mean value of $x$ on region $k$

• $g_k$ is the $X$-gradient in region $k$

• $h_k$ is the $Z$-gradient in region $k$

• $X_i$ is the $X$-coordinate of the centre of pixel $i$

• $Z_i$ is the $Z$-coordinate of the centre of pixel $i$

• $\bar{X}_k$ is the $X$-coordinate of the centre of gravity of region $k$

• $\bar{Z}_k$ is the $Z$-coordinate of the centre of gravity of region $k$

Thus

$$
\bar{X}_k = \frac{\sum_{i \in R_k} X_i}{\sum_{i \in R_k} 1}, \quad \bar{Z}_k = \frac{\sum_{i \in R_k} Z_i}{\sum_{i \in R_k} 1}.
$$

To express the value of the pixels $x$ in terms of the mean values $z$ and gradients $h$ and $g$, a matrix $G$ is introduced such that

$$
x = Gw
$$

where $w = [z, g, h]^T$. $G$ has the form $G = [F C D]$ where

$$
C_{ik} = \begin{cases} 
X_i - \bar{X}_k & \text{pixel } i \text{ in region } k \\
0 & \text{otherwise}
\end{cases}
$$

and

$$
D_{ik} = \begin{cases} 
Z_i - \bar{Z}_k & \text{pixel } i \text{ in region } k \\
0 & \text{otherwise}
\end{cases}
$$

The quantities in each region are assumed independent, and the means are assumed independent of the gradients. To account for prior knowledge of the direction of diffusions and flows or other phenomena that produce gradients in a region, it is assumed that $(g_i, h_i)$ are Gaussian. A prior value (probably $(0, 0)$) and a confidence ellipsoid in $\mathbb{R}^2$ that characterises the prior assumptions about the gradient in region $i$ is then given.

To estimate the mean values and gradients from measurements, equation $y = AGw$ is used, and the process continues as usual. The MMSE process will estimate the mean values in each region, and the gradient in each region.
4.5 Signal, noise and information content

The singular value decomposition (SVD, see Appendix D) can give insight into the information content of the measurements, as judged by the estimation error covariance $\Sigma_{\text{est}}$ compared to the prior covariance $\Sigma_x$. A transformed $A$ matrix is defined as:

$$\tilde{A} = \Sigma_v^{-1/2} A \Sigma_x^{-1/2},$$

(4.2)

This is a version of $A$ with its input and output scaled so that both have covariance $I$. The matrix $\tilde{A}$ can be directly interpreted as giving the signal-to-noise ratio of the measurement, which gives an idea of its value in the estimation.

To understand the information content in $\tilde{A}$, the following transformations are performed:

Express equation 4.2 by $A$,

$$A = \Sigma_v^{1/2} \tilde{A} \Sigma_x^{-1/2},$$

(4.3)

using the equation for the generic linear model

$$y = Ax + v$$

(4.4)

and substituting $A$ in equation 4.4 by equation 4.3:

$$y = \Sigma_v^{1/2} \tilde{A} \Sigma_x^{-1/2} x + v$$

$$\Sigma_v^{-1/2} y = \tilde{A} \Sigma_x^{-1/2} x + \Sigma_v^{-1/2} v$$

(4.5)

Define:

$$\hat{x} = \Sigma_x^{-1/2} x$$

$$\hat{v} = \Sigma_v^{-1/2} v$$

The measurement vector $y$ is scaled with the same matrix, $\Sigma_v^{-1/2}$, as the noise vector, $v$. Since $x$ and $v$ are Gaussian, $\hat{x}$ and $\hat{v}$ are Gaussian with expected value of 0 and covariance $I$. Thus $\hat{x}$ is the normalised input vector, and $\hat{v}$ is the normalised noise vector.

Denote the singular values of $\tilde{A}$, $\hat{\sigma}_i$, where

$$\hat{\sigma}_1 \geq \hat{\sigma}_2 \geq \ldots \geq \hat{\sigma}_r > 0$$

and $r$ is the rank of $\tilde{A}$.

The following interesting and useful conclusions can be made:

- The number of singular values of $\tilde{A}$ that exceed approximately 1, is the number of dimensions ($s$) along which the measurement is useful.
• The input directions \( v_1, \ldots, v_s \) associated with these "substantial" singular values, give the subspace along which the measurement provides useful information.

To investigate the information content in the magnetic measurements, the \( A \)-matrices for four different models are generated. All models cover 2400m vertically and 3000m horizontally, with the measurements being made 200m above the top layer. Inclination is set to 75°, Declination=0° and Azimuth=0°. The total ambient field is set to 50000 nT. The reliabilities for the pixels, \( \sigma_i \), are set to \( 0.2 \times 10^{-3} \) SI — units and assumed noise in the measurements, \( \sigma_v = 0.1 \) nT. The geometries of the four models are:

- i) model divided into \( 6 \times 15 \) pixels with 31 measurements (\( \Delta y = 100 \) meters)
- ii) model divided into \( 6 \times 15 \) pixels with 61 measurements (\( \Delta y = 50 \) meters)
- iii) model divided into \( 12 \times 30 \) pixels with 31 measurements (\( \Delta y = 100 \) meters)
- iv) model divided into \( 12 \times 30 \) pixels with 61 measurements (\( \Delta y = 50 \) meters)

Figure 4.4 a shows the singular values of \( \tilde{A} \) for these four models. The red and magenta curves represents the low pixel resolution models, i and ii, the blue and the black curves represents the higher pixel resolution models, iii and iv.

By examining the singular values of \( \tilde{A} \) for the various models the following observations can be made:

- For low resolution models (i and ii ), the curve drops below 1 at approximately 10 measurements.
- Above 15 measurements the contribution drops rapidly to 0.
- For high resolution models (iii and iv), the curve drops below 1 at approximately 7 – 9.
- Above 15 measurements the curve drops to 0 more smoothly.

Figure 4.4 b plots the singular values of \( \tilde{A} \) for three models with various reliabilities and sensor noise, using the geometries of the low resolution model, i. Using \( \sigma_i = 0.2 \times 10^{-3} \) SI — units and \( \sigma_v = 0.1 \) nT, the (red) curve drops below 1 at roughly 9 dimensions, while improved accuracy in the measurements, \( \sigma_v = 0.05 \) nT indicates approximately 13 useful measurements (magenta curve). With decreasing (prior) reliabilities, \( \sigma_i = 0.5 \times 10^{-3} \) SI — units, the number of meaningful measurements (~14) increases.

Studying the singular values of \( \tilde{A} \) for these models, the following conclusions can be made:
• The resolution in the model (number of pixels) has some influence on the measurements.

• The number of meaningful measurements is $10 - 15$, i.e., the 31 or 61 measurements could be substituted by $10 - 15$ independent measurements.

• Reduced sensor noise improves the estimation.

• The improvement in estimation is greater when the prior reliabilities are low.

4.6 Forward model examples

Examples of calculated anomalies

The calculated magnetic anomalies of a simple geometrical shape are shown below. The model in Figure 4.5 a is used to demonstrate the effect of different inclinations and azimuths, see Figures 4.5 b, 4.5 c and 4.5 d. For an inclination of $90^\circ$, the magnetic anomaly will be the same for all azimuths. For other inclinations, the shape of the anomalies will be different for various azimuths.

Detection limits for magnetic anomalies

To demonstrate the effects of the geometry of the model related to the measurement, one column of the A-matrix representing the contribution from each pixel to the measurement at a certain $x$-location, is visualised as a colour image (see figure 4.6 a). Specifications for this model: 10 pixels (vertically) $\times$ 15 pixels (horizontally) with each pixel representing $200 \times 200$m. The ambient magnetic field is set to 50000 nT, inclination is $75^\circ$, declination and azimuth is $0^\circ$. Each pixel is assigned a susceptibility of $0.4 \times 10^{-3}$ SI – units. The measurement is above pixel (1.8) ($x$-location 1500m) with measurement-height at 200m above the upper layer. In Figure 4.6 b the contribution (in nT) from each pixel along 4 horizontal lines are shown. Figure 4.6 c shows the contribution from 4 vertical lines.

The effect from each pixel to the measurement significantly decreases as the distance increases. Studying Figure 4.6 b, the uppermost line (red) has a significant contribution, both as a positive and a negative anomaly. The black curve (at 1600-1800 meters depth) shows minor contribution from this depth, since the sensor has a typical resolution of 0.1 nT. It is, however, important to take into account that the measurement is an accumulative result from each pixel. A sum of small numbers could still be above the resolution threshold of the sensor, but it will not result in a high frequency anomaly. Figure 4.6 c illustrates the effect of depth related to the distance from the sensor. The red curve shows the contribution from each pixel directly beneath the sensor, and it can easily see how fast the values drop as a function
of depth. At about 1200 meters depth the contribution from each pixel is less than the sensor resolution.

**Edge effects**

As shown in the previous subsection, a measurement is influenced by the susceptibilities of the pixels beneath and to both sides of the measurement point. When performing a forward calculation of the anomalies, given a susceptibility model, it is important to expand the model somewhat to avoid unrealistic anomalies towards the ends of the profile, called *end effects*. The edge effects are caused by no data within the influence area of the measurements. To study the edge effects to the measurements, along a measurement profile, to the measurements, a model with $20 \times 20$ pixels, each representing $200 \times 200$ meters is generated. The measurement is taken at altitude 200 meter above the leftmost column ($x$ location = 0m). If it is assumed that the susceptibilities in all pixels to be $0.4 \times 10^{-3}$ SI – units, the measurement would have been 2.25 nT. The image in Figure 4.7a shows the contribution from each pixel to the measurement (2.25 nT) at $x$-location 0 colour-coded; values ranging from −0.164 to 0.955 nT. The contributions are dominated by the pixels in the uppermost layers as well as below the measurement point. In figure 4.7b the sum of contributions (in nT) for each column are plotted. The shape of the curve goes horizontal at approximately 2000 meters, indicating an estimation with these specifications need an expansion of the model by approximately 10 columns on each side of the study area; beyond this, the pixels have very little influence to the measurements.

In a real modelling, it is important to choose aeromagnetic measurement profiles which is not biased by cultural effects from e.g., wells, platforms or pipelines. This method can be used to estimate the effects of such installations (with extremely high susceptibilities) to the measurements by e.g., including the geometry and susceptibility of a well. By studying the detection limits for this model, a decision can be made of how close to the installations a reasonable inversion can be performed. A discussion of detection limits and end effects has to be performed for each real case study.
Figure 4.4: a) Plot of singular values of $\tilde{A}$ for two different pixel resolutions and two different measurement densities: $6 \times 15$ pixels, 31 measurements (red), $6 \times 15$ pixels, 61 measurements (magenta), $12 \times 30$ pixels, 31 measurements (blue) and $12 \times 30$ pixels, 61 measurements (black). b) Plot of singular values of $\tilde{A}$ for $6 \times 15$ pixels, 31 measurements with $\sigma_i = 0.2 \times 10^{-3}$ SI units, $\sigma_v = 0.1$ nT (red); $\sigma_i = 0.2 \times 10^{-3}$ SI units, $\sigma_v = 0.05$ nT (magenta); $\sigma_i = 0.5 \times 10^{-3}$ SI units, $\sigma_v = 0.1$ nT (blue). Logarithmic vertical axes.
Figure 4.5: a) A 5 x 30 pixel model. Each pixel represents 200 x 200m. The “box-pixels” are assigned susc. = 0.2 x 10^{-3} SI units, the other pixels are assigned susc. = 0. Measurement height=200m; sampling from 1500m-4500m, with 100m steps. Ambient magnetic field is assumed 50000 nT. b) Calculated magnetic anomaly, for inclination=90°, for various azimuths (note that this gives the same anomaly), c) inclination=75° (azimuth 0°=blue, 45°=red, 90°=black), d) inclination=45° (azimuth 0°=blue, 45°=red, 90°=black).
**Figure 4.6:** a) The A-matrix for a measurement at x-location 1500m, measured at 200m height, inclination=75°, declination=0° and azimuth=0°; assumed susceptibilities are $0.4 \times 10^{-3}$ SI – units for all pixels. A colour representation showing the contribution from each pixel to the measurement (4.63 nT). The horizontal lines at row (1,3,6,9) and the vertical lines at columns (8,10,12,14) are shown. b) Profiles showing the contributions to the measurements from four horizontal lines (through 4 depths), row-1 (red), row-3 (blue), row-6 (magenta), row-9 (black). c) The contributions from four vertical lines, column-8 (red), column-10 (blue), column-12 (magenta), column-14 (black).
Figure 4.7: a) Plot of A-matrix with susceptibilities $0.4 \times 10^{-3}$ SI units assigned to each pixel; the model has $10 \times 15$ pixels, each representing $200 \text{m} \times 200 \text{m}$, inclination=$75^\circ$, declination=$0^\circ$, azimuth=$90^\circ$, ambient field = 50000 nT. The measurement (2.25 nT) is made at $x$-coordinate 0m, altitude 200m. b) The sum of contributions (in nT) from each column in the model, to the measurement.
Chapter 5

MMSE for Magnetic Anomaly Interpretation

As presented in the previous chapters, several approaches have been undertaken in an attempt to deal with the non-uniqueness and the relatively small amount of information inherent in the magnetic measurements. Using airborne magnetic data in geological mapping of frontier areas, these approaches might be the only possibility, due to the available prior geological information being very limited. When used in exploration for oil/gas in more mature areas, there will exist geological models based on interpreted seismic. These models are the basis for generating prior models which consists of stratigraphic layers and faults with susceptibilities determined from well measurements and general knowledge. The quality or uncertainty of the prior geological models varies, primarily based on the quality of the seismic data and the complexity of the subsurface. By introducing a statistical framework, in addition to the model geometry, it is possible to optimally combine prior information with magnetic measurements, see figure 5.1. This inversion approach should result in more accurately estimated susceptibilities based on susceptibilities from the prior model and airborne magnetic measurements. The use of uncertainties from the prior model is an important element in this method, since the uncertainty in the "best guess" will influence the corrections made by the inversion.

The goal of this inversion is to use magnetic data, which is independent information compared to seismic, in order to tune the prior model. Since the geometry of the model will be fixed, only the magnetic properties will be changed, an estimated model will be used in an iterative fashion combined with further interpretation of the seismic data. Different hypotheses which may explain the mismatch between measured and forward modelled magnetic anomalies can be introduced to test for their reliabilities and judge their geological relevance.

The estimation method may be applied on various models, e.g., models for implementing prior knowledge. In the following sections, a variety of such models, with increasing degree of prior information, will be tested and then compared.
Figure 5.1: The steps in magnetic inversion using MMSE estimation: The prior geometry is derived from a seismic cross-section, prior susceptibilities and reliabilities are assigned by the geologist. The information in the magnetic measurements blend with the prior model, resulting an estimated susceptibility model with improved reliabilities.
5.1 The reference susceptibility model

In this chapter, several different estimators will be applied to the same measurements which come from the “reference” model described in this section.

The geometry is shown below in figure 5.2. The pixel geometry is 10 (in Z-direction) × 44 (in X-direction), each pixel is 200m × 400m. Thus, the entire region considered is 8800m wide, and 4000m deep.

To take care of edge effects (see Section 4.6) the model is expanded 10 columns on each side compared to the calculated magnetic anomalies. The estimations in the following examples are based on the 10 × 44 pixels model, but all plots are limited to 10 × 24 pixels (skipping 10 pixels to the left and right).

It consists of six regions, labelled 1-6, each of these regions has a different average susceptibility, as well as a gradient of susceptibility. The susceptibilities of the regions are defined in table 5.1.

In layer-2, a horizontal gradient of $0.1 \times 10^{-3}$ SI – units per 8800 meters is definier. The left fault is assigned a vertical gradient of $-0.1 \times 10^{-3}$ SI – units per 2800 meters, i.e., increasing susceptibilities upwards. Gradients like these may occur in a real model due to some migration of oil or gas.
<table>
<thead>
<tr>
<th>Region</th>
<th>min((\tilde{z}))</th>
<th>max((\tilde{z}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. layer-1</td>
<td>0.20</td>
<td>0.20</td>
</tr>
<tr>
<td>2. layer-2</td>
<td>0.25</td>
<td>0.35</td>
</tr>
<tr>
<td>3. layer-3</td>
<td>0.40</td>
<td>0.40</td>
</tr>
<tr>
<td>4. left fault</td>
<td>0.45</td>
<td>0.54</td>
</tr>
<tr>
<td>5. right fault</td>
<td>0.60</td>
<td>0.60</td>
</tr>
<tr>
<td>6. basement</td>
<td>1.00</td>
<td>1.00</td>
</tr>
</tbody>
</table>

Table 5.1: Susceptibilities for each region, in \(10^{-3}\) SI \(\text{units}\). Layer-2 has a horizontal gradient of \(0.1 \times 10^{-3}\) SI \(\text{units per } 8800\text{m}\), and the left fault has a vertical gradient of \(-0.1 \times 10^{-3}\) SI \(\text{units per } 2800\text{m}\).

Measurements

The "measurements" assume the following:

- Inclination = 72°,
- Declination = 0°,
- Azimuth = 0°
- Magnitude of ambient earth field, \(H_0 = 52000\text{ nT}\)

The measurements, \(y\), are taken at a height of 200m, spaced 100m apart, starting at \(X = 2000\text{m}\) and going to \(X = 6800\text{m}\). (Thus the edge effects is to a great extent avoided.) All together there are 49 magnetic measurements. The measurements resulting from true \(x\) vector, without measurement noise, is shown in figure 5.3 a.

The measurement noise is zero mean white Gaussian (i.e., \(\nu_i\) and \(\nu_j\) are zero mean, uncorrelated, and Gaussian). The standard deviation of the measurement noise is taken to be 0.2 nT, which is typical for a state-of-the-art magnetic survey. The simulated noise and the measurement with its noise are shown in figures 5.3 a and b.

All of the estimators developed in this chapter will be tested on this particular measurement signal \(y\). The different estimators arise from different prior assumptions about \(x\).

5.2 Uncorrelated pixel model

The simplest model assumes no correlation between any pixels. In other words, it is assumed that \(\Sigma_x\) is diagonal. Since the susceptibilities typically lie in the range 0 to
Figure 5.3: a) The anomalies for the true susceptibility model without noise, b) random Gaussian noise with std.dev. 0.2 nT, c) measurements including Gaussian white noise.
$1 \times 10^{-3}$ SI — units, the prior mean of each pixel is assumed $0.5 \times 10^{-3}$ SI — units, and the standard deviation is assumed $0.5 \times 10^{-3}$ SI — units. Thus:

$$\Sigma_x = (0.5 \times 10^{-3})^2 I$$

where $I$ is a $440 \times 440$ identity matrix.

The resulting estimate is shown in figure 5.4 b, and the absolute estimation error, $|\hat{x}|$, for each pixel is shown in c. Evidently the estimate is not very good, but it does predict that something is happening near the two fault locations, at least near the surface. The poor performance of this estimator is expected, since we use very little prior information. This example shows that simple inversion is essentially impossible.

The forward fit obtained with the estimate is shown in figure 5.4 a along with the actual measurements. The RMS error, $\tau = 0.02$, the same order of magnitude as the sensor noise. Thus, the forward fit for this estimate is excellent. Once again, a good forward fit does not imply good estimation, in fact figures 5.4 b and 5.2, which are very different, results in the same forward calculated anomalies (within the noise level)! This clearly demonstrates the non-uniqueness in magnetic data.

The reliability measure $\lambda_i$ for each pixel is shown in figure 5.4 d. It shows that confidence in the estimated values is approximately the same as for the prior values, i.e., the experiment is not very informative. In fact the information measure $\eta$ is equal to 0.97, which shows the information value of the measurements is essentially zero. Note that the reliability is slightly better in a few pixels near the surface.

### 5.3 Spatial pixel correlation model

Here the simplest model of pixel correlation is used, i.e., distance correlation. The prior mean of and standard deviation for each pixel are the same. In the first example a distance correlation of $800$m vertically and $2000$m horizontally is used, see figure 5.5 a. This simply captures the fact that adjacent pixels are likely to have not so different values. In fact, it is not a particularly good model, e.g., the susceptibility changes rapidly at the faults.

The resulting estimate is shown in figure 5.6 b, and the absolute estimation error, $|\hat{x}|$, for each pixel is shown in c. The estimate is rather poor, however, it does predict anomalies close to the two fault locations. It is slightly better than the estimate with no correlation. The performance of this estimator is poor, since very little prior information is used.

The forward fit obtained with the estimate is shown in figure 5.6 a along with the actual measurements. The RMS error ($\tau = 0.09$) is not as good as the uncorrelated model.

The reliability measure $\lambda_i$ for each pixel is shown in figure 5.6 d. It shows that confidence in the estimated values is slightly improved over a larger area in the
Figure 5.5: The images show how pixel (5,12) at 2000m vertically and 4400m horizontally correlates to the surrounding pixels, a) 800m × 2000m decorrelation distance, b) 2000m × 800m decorrelation distance.
Figure 5.6: a) The forward calculated estimated model (in blue) compared to the measurements (in red), b) estimated model based on decorrelation distance 800m \times 2000m, c) the absolute estimation error, |\hat{\xi}|, for each pixel, d) reliability ratio, \lambda_i, for each pixel. (\times 10^{-3} \text{ SI} – units in b-c.)
model. The information measure $\eta$ is equal to 0.85, which shows that using the measurements has reduced the overall uncertainty by 15%.

By selecting a distance correlation matrix of 2000m $\times$ 800m (see figure 5.5 b), a better forward fit of $\tau = 0.05$ is achieved (see figure 5.7 a).

The resulting estimate is shown in figure 5.7 b, and the absolute estimation error, $|\tilde{x}|$, is shown in c. The estimate is still not very good, but it does predict the two fault good, while the layers are more or less ignored. The reliability measure $\lambda_i$ for each pixel is shown in figure 5.7 d. It shows that confidence in the estimated values is improved significantly in the upper part of the model. The information measure $\eta$ is equal to 0.79, which shows that using the measurements has reduced the overall uncertainty with 21%.

### 5.4 Region model

In this section a large amount of prior information is incorporated, namely various regions. The model assumes that the susceptibility is constant within each region, between regions there is no correlation. The pixels all have the same prior mean $0.5 \times 10^{-3}$ SI units, and the same standard deviation, $0.5 \times 10^{-3}$ SI units. They have the following correlation coefficients: if two pixels are in the same region, the correlation coefficient is one; if two pixels are in different regions, the correlation coefficient is zero. This example is meant to show what happens when prior information, say from seismic analysis well logs, is incorporated into the estimation process. Note that the prior information, though useful, is not in fact accurate: some of the regions $x$ have gradients, which are not reflected in this model. This assumes that $x$ is constant in each region. The estimation of magnetic properties for the regions are calculated according to the following equation:

$$\tilde{z} = E(z|y) = \tilde{z} + B(y - \bar{y}) \quad (5.1)$$

where

$$\bar{y} = AF\tilde{z} + \bar{v}$$

is the expected value of the sensor measurement $y$, $B$ is defined by equation 3.25 and $F$ is explained in Section 4.4.

The resulting estimate is shown in figure 5.8 b. The estimate is good, the susceptibilities for the regions are close to the exact model, see table 5.2.

Figure 5.8 c shows the absolute estimation error, $|\tilde{x}|$, which in general is low. The gradients in the exact model in layer-2 and left fault is reflected in the estimated model.

The overall improvement in reliability ($\eta = 0.30$) implies that the measurement has significantly contributed to the improved confidence in the estimated values. The forward fit (see figure 5.8 a) is excellent ($\tau = 0.03$) and lies within the noise.
Figure 5.7: a) The forward calculated estimated model (in blue) compared to the measurements (in red), b) estimated model based on decorrelation distance 2000m x 800m, c) the absolute estimation error, $|\hat{z}|$, for each pixel, d) reliability ratio, $\lambda_i$, for each pixel. ($x 10^{-3}$ SI – units in b-c.)
Figure 5.8: a) The forward calculated estimated model (in blue) compared to the measurements (in red), b) estimated model based on the region model, c) the absolute estimation error, $|\mathbf{z}|$, for each pixel, d) reliability ratio, $\lambda$, for each pixel. ($\times 10^{-3}$ SI - units in b-c.)
5.5. REGIONS WITH GRADIENTS MODEL

<table>
<thead>
<tr>
<th>Region</th>
<th>$\bar{z}$</th>
<th>$\hat{z}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. layer-1</td>
<td>0.20</td>
<td>0.20</td>
</tr>
<tr>
<td>2. layer-2</td>
<td>0.30</td>
<td>0.05</td>
</tr>
<tr>
<td>3. layer-3</td>
<td>0.40</td>
<td>0.41</td>
</tr>
<tr>
<td>4. left fault</td>
<td>0.50</td>
<td>0.55</td>
</tr>
<tr>
<td>5. right fault</td>
<td>0.60</td>
<td>0.58</td>
</tr>
<tr>
<td>6. basement</td>
<td>1.00</td>
<td>1.10</td>
</tr>
</tbody>
</table>

Table 5.2: Susceptibilities for each region, in $10^{-3}$ SI — units, prior ($\bar{z}$) and estimated ($\hat{z}$), for the region model.

limit. The uncertainty reduction, $\lambda$ is plotted in figure 5.8 d. It shows a significant improvement in reliabilities of 50 — 80%.

5.5 Regions with gradients model

The estimation of magnetic properties for the regions, included gradients, are calculated according to the following equation:

$$\hat{w} = E(w|y) = \bar{w} + B(y - \bar{y})$$

where

$$\bar{y} = AG\bar{w} + \bar{v}$$

is the expected value of the sensor measurement $y$, $B$ is defined in equation 3.25 and $G$ is explained in Section 4.4.

In this model, the following assumptions are encoded:

- all pixels have a prior mean $0.5 \times 10^{-3}$ SI — units, and a standard deviation, $0.5 \times 10^{-3}$ SI — units
- correlation coefficients: if two pixels are in the same region, the correlation coefficient is one; if two pixels are in different regions, the correlation coefficient is zero
- there might be a horizontal gradient in layer-2 of magnitude $0.1 \times 10^{-3}$ SI — units per 8800m and vertical gradients in the left fault of magnitude $-0.1 \times 10^{-3}$ SI — units per 2800m; no gradient is encoded in the other regions

The resulting estimate is shown in figure 5.9 b. The estimate is good, the susceptibilities for the regions are close to the true model, see table 5.3. A gradient is estimated for layer-2, but the values are in general to low, and some are negative
Figure 5.9: a) The forward calculated estimated model (in blue) compared to the measurements (in red), b) estimated model based on the region model with gradients, c) the absolute estimation error, $|\tilde{z}|$, for each pixel, d) reliability ratio, $\lambda_i$, for each pixel. ($\times 10^{-3}$ SI – units in b-c.)
5.6. DETAILED PRIOR INFORMATION MODEL

<table>
<thead>
<tr>
<th>Region</th>
<th>min((\hat{z}))</th>
<th>max((\hat{z}))</th>
<th>min((\hat{z}))</th>
<th>max((\hat{z}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. layer-1</td>
<td>0.200</td>
<td>0.200</td>
<td>0.218</td>
<td>0.218</td>
</tr>
<tr>
<td>2. layer-2</td>
<td>0.250</td>
<td>0.350</td>
<td>-0.026</td>
<td>0.026</td>
</tr>
<tr>
<td>3. layer-3</td>
<td>0.400</td>
<td>0.400</td>
<td>0.438</td>
<td>0.438</td>
</tr>
<tr>
<td>4. left fault</td>
<td>0.450</td>
<td>0.540</td>
<td>0.438</td>
<td>0.563</td>
</tr>
<tr>
<td>5. right fault</td>
<td>0.600</td>
<td>0.600</td>
<td>0.592</td>
<td>0.592</td>
</tr>
<tr>
<td>6. basement</td>
<td>1.000</td>
<td>1.000</td>
<td>1.057</td>
<td>1.057</td>
</tr>
</tbody>
</table>

Table 5.3: The range of susceptibilities for each region, in \(10^{-3}\) SI – units, prior (\(\hat{z}\)) and estimated (\(\hat{z}\)), for the gradient model.

(which is not likely). A gradient is estimated for the left fault, and the estimated values are very close to the reference model.

Figure 5.9 c shows the absolute estimation error, \(|\hat{z}|\), which in general is low.

The overall improvement in reliability, \(\eta = 0.30\) implies that the measurement has significantly contributed to an increased confidence in the estimated values. The forward fit, shown in figure 5.9 a, is excellent (\(\tau = 0.03\) and lies within the noise limit. The uncertainty reduction, \(\lambda\) is plotted in figure 5.9 d, shows that the improvement in reliabilities varies from approximately 30 – 70\%, which is significant. The lowest improvement in reliabilities is for layer-2, while the other layers and the faults have relatively high improvement in their reliabilities.

## 5.6 Detailed prior information model

In this model a significant amount of detailed prior information has been incorporated:

- the region geometries are known
- the susceptibilities of the horizontal layers, including the basement, are known (within \(\pm 0.1 \times 10^{-3}\) SI – units)
- the susceptibility of the left fault is known (within \(\pm 0.3 \times 10^{-3}\) SI – units)
- the susceptibility of the right fault is known (within \(\pm 0.4 \times 10^{-3}\) SI – units)
- layer-2 is encoded with a horizontal gradient of \(0.1 \times 10^{-3}\) SI – units per 8800m
- left fault is encoded with a vertical gradient of \(-0.1 \times 10^{-3}\) SI – units per 2800m

The resulting estimate is shown in figure 5.10 b. The estimate is good, the susceptibilities for the regions are close to the exact model, see table 5.4.
Figure 5.10: a) The forward calculated estimated model (in blue) compared to the measurements (in red), b) estimated model based on the detailed prior information, c) the absolute estimation error ($|\hat{x}|$) for each pixel, d) reliability ratio, $\lambda_i$, for each pixel. ($\times 10^{-3}$ SI - units in b-c.)
Table 5.4: The range of susceptibilities for each region, in $10^{-3}$ SI – units, prior ($\hat{z}$) and estimated (\bar{z}), for the detailed prior information model.

<table>
<thead>
<tr>
<th>Region</th>
<th>min($\hat{z}$)</th>
<th>max($\hat{z}$)</th>
<th>min($\bar{z}$)</th>
<th>max($\bar{z}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. layer-1</td>
<td>0.20</td>
<td>0.20</td>
<td>0.245</td>
<td>0.245</td>
</tr>
<tr>
<td>2. layer-2</td>
<td>0.25</td>
<td>0.35</td>
<td>0.197</td>
<td>0.233</td>
</tr>
<tr>
<td>3. layer-3</td>
<td>0.40</td>
<td>0.40</td>
<td>0.383</td>
<td>0.383</td>
</tr>
<tr>
<td>4. fault/left</td>
<td>0.45</td>
<td>0.50</td>
<td>0.500</td>
<td>0.603</td>
</tr>
<tr>
<td>5. fault/right</td>
<td>0.60</td>
<td>0.60</td>
<td>0.638</td>
<td>0.638</td>
</tr>
<tr>
<td>6. basement</td>
<td>1.00</td>
<td>1.00</td>
<td>1.000</td>
<td>1.000</td>
</tr>
</tbody>
</table>

Figure 5.10 c shows the absolute estimation error, $|\hat{z}|$, which in general is low.

The overall improvement in reliability ($\eta = 0.11$) implies a very good estimation. The forward fit (figure 5.10 c), is excellent ($\tau = 0.03$) and lies within the noise limit. The uncertainty reduction, $\lambda$ is plotted in figure 5.10 d, and shows that the improvement in reliabilities is better than 85% for all regions, which is very good. This model gives by far the best estimation combined with a perfect forward fit.

### 5.7 Summary and comparison

In all of the above modelling examples the ambiguity of magnetic data is clearly demonstrated: all the estimated models (which are quit different - especially the pixel models), result in the same forward anomalies. Table 5.5 shows the improvement in reliabilities($\eta$) for all cases, and the measure for forward fit, $\tau$.

<table>
<thead>
<tr>
<th>model</th>
<th>$\eta$</th>
<th>$\tau$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uncorrelated pixels</td>
<td>0.97</td>
<td>0.02</td>
</tr>
<tr>
<td>Distance correlation</td>
<td>0.85</td>
<td>0.09</td>
</tr>
<tr>
<td>Distance correlation</td>
<td>0.79</td>
<td>0.04</td>
</tr>
<tr>
<td>Regions with constant susc.</td>
<td>0.30</td>
<td>0.03</td>
</tr>
<tr>
<td>Regions with gradients</td>
<td>0.30</td>
<td>0.03</td>
</tr>
<tr>
<td>Regions/grad. with detailed info</td>
<td>0.11</td>
<td>0.03</td>
</tr>
</tbody>
</table>

Table 5.5: The overall reliability improvement, $\eta$, and the forward fit, $\tau$, for all the tested models.

Except for the “detailed prior information” model, very little prior information is given to the estimation process. The pixels all have the same prior mean $0.5 \times 10^{-3}$ SI – units, and the same standard deviation, $0.5 \times 10^{-3}$ SI – units, which mean they all can vary roughly between 0 and $1 \times 10^{-3}$ SI – units. In the “uncorrelated
pixel model", the estimated model is very poor compared to the "reference model". When applying spatial correlation, some features in the reference model emerges, but mainly in the top pixel-rows. In the region model, the prior model is simplified; the large number of pixels are mapped into 6 regions. The result is quit promising, the parameters have an acceptable correspondence, and the overall improvement in reliabilities are significant. The encoding of gradients in some regions result in a similar overall improvement, and the left fault is very well estimated. When detailed information is given, the estimated values are very close to the correct values (given by the reference model), and the overall improvement is very good. In a real case, the degree of prior information could be close to what is used in this example.

The main conclusions from these tests are:

- forward fit is good for all estimates, in spite of the fact that some estimates are very bad and some are very good
- the reliability measures are accurate in the sense they inform the interpreter when an estimate is valuable
- using a lot of prior information may give a very good estimation
- a good forward fit does not necessarily mean the estimated model is correct

The rest of this chapter will comment on the robustness of the MMSE estimation if the prior model geometry is wrong and demonstrate how the prior information is blended with the measurements. Finally, the use of this novel method for survey planning and evaluation of magnetic measurements will be discussed.

5.8 Incorrect prior information

In the previous sections, true (correct) regions were used. The robustness of the method is now tested by using two examples of incorrect geometry of regions in the prior models. Figure 5.11 a shows the "correct" region model with no gradient encoded, b and c two "incorrect" region model geometries; one with an error in layer-2 (i), one with truncated right fault (ii).

The point here is that when prior information is used, it is of course possible that it is wrong. The question is whether slightly incorrect prior information can have a huge effect on the resulting estimate.

The estimated models for each are shown in figures 5.12 a - c. Table 5.6 lists the estimated values, \( z \), for each model. The error (i) has created some changes to the estimated susceptibilities in layer-2, layer-3 and basement. The error (ii) results in some changes in estimated susceptibilities in layer-2 and basement. All "errors" are compared to the estimated "correct" model, and they are found to be relatively small. Thus the conclusion can be made that slightly wrong prior information does not have great impact on the estimation.
Figure 5.11: a) The "correct" region model with assigned susceptibilities, b) model with an "error" in the region for layer-2, c) model with an "error" in the region of the right fault. (× 10^{-3} SI - units in a-c.)
Figure 5.12: a) The estimated "correct" region model, b) the estimated model with prior "error" in the region for layer-2, c) the estimated model with prior "error" in the region of the right fault. (× 10⁻³ SI — units in a-c.)
<table>
<thead>
<tr>
<th>Region</th>
<th>correct $\hat{z}$</th>
<th>wrong layer $\hat{z}$</th>
<th>wrong fault $\hat{z}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. layer-1</td>
<td>0.20</td>
<td>0.21</td>
<td>0.21</td>
</tr>
<tr>
<td>2. layer-2</td>
<td>0.30</td>
<td>0.24</td>
<td>0.17</td>
</tr>
<tr>
<td>3. layer-3</td>
<td>0.40</td>
<td>0.39</td>
<td>0.40</td>
</tr>
<tr>
<td>4. left fault</td>
<td>0.50</td>
<td>0.54</td>
<td>0.53</td>
</tr>
<tr>
<td>5. right fault</td>
<td>0.60</td>
<td>0.60</td>
<td>0.60</td>
</tr>
<tr>
<td>6. basement</td>
<td>1.00</td>
<td>1.04</td>
<td>1.07</td>
</tr>
</tbody>
</table>

Table 5.6: Susceptibilities for each region, in $10^{-3}$ SI – units, for the prior “correct” model and the estimated susceptibilities for the two “wrong” prior models.

5.9 Blending prior information with measurements

In this section it is focused on how the blending of prior information with the magnetic measurements is performed. The estimated $\hat{z}$ is calculated by this equation (ref equation 5.1):

$$\hat{z} = \mathbb{E}(z|y) = \bar{z} + B(y - \bar{y}),$$

where:

$$B = \Sigma_x A^T (A \Sigma_x A^T + \Sigma_y)^{-1},$$

see Section 3.6.

The linear correction is based on the discrepancy $(y - \bar{y})$, which is the difference between the actual sensor measurement $(y)$ and the expected sensor measurement $(\bar{y})$. $A \Sigma_x A^T$ can be interpreted as the “signal covariance” and $\Sigma_y$ is the noise covariance. The estimator matrix $B$ (also denoted “gain”) depends on the “signal to noise ratio” at the sensor; if $B$ is large, the signal-to-noise ratio is high, and small when it is low.

To demonstrate the estimation, model (i) in the previous section (model with “error” in layer-2) is selected. The discrepancy in the measurements are plotted in figure 5.13 a. This discrepancy is used to blend the prior information with the measurements using $B$ as a correction factor for each region. In this way, the prior guess of susceptibilities $\hat{z}$ for each region is modified by the discrepancy in each measurement multiplied by the gain given by $B$ (see figure 5.13 b).

5.10 Survey planning and evaluation

Measurement density

To investigate the effect of the sampling rate related to resolution in the measurements, the model in Section 5.1 has used for testing sample spacing, $\Delta y$, of 25, 50,
Figure 5.13: a) The difference between the actual measurements and the expected measurements (discrepancy), b) the correction factors \((B)\) for each measurement \((49)\), per region: region-1 (black), region-2 (green), region-3 (cyan), region-4 (blue), region-5 (magenta), region-6 (red).
100, 200, 300 and 400 meters. The number of measurements with 25 meter resolution, $\Delta y_{25}$, are $n_{25} = 193$. For $\Delta y_{100}$, $n_{100} = 49$, and for $\Delta y_{400}$, $n_{400} = 13$. To compare the different resolutions, the measurements for the various densities are interpolated to 193 samples, denoted $y_i$. The forward calculated magnetic anomalies for each of these sampling densities are plotted in figure 5.14.

Visually it can be observed that sampling densities of 300m and 400m deviates significantly from the 25m resolution.

To calculate the forward fit of the various measurement densities compared to the 25m density, the following measure is used:

$$r = \frac{||y_i - y_{25}||}{||y_{25}||}$$  \hfill (5.2)

By inspecting table 5.7 note that a sampling rates of 300m and 400m have average error above the sensor noise level of 0.2 nT. Sampling rates between 200m and 300m should be sufficient with this geometry and flight height of 200m. The total distance from the first to the last measurement in this example is 4800m, indicating that only 15 – 20 measurements are required.
Table 5.7: Average error (in nT), compared to 25m sampling, for various measurement densities (in meters).

<table>
<thead>
<tr>
<th>$\Delta y_i$</th>
<th>$\tau_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>0.001</td>
</tr>
<tr>
<td>100</td>
<td>0.004</td>
</tr>
<tr>
<td>200</td>
<td>0.016</td>
</tr>
<tr>
<td>300</td>
<td>0.261</td>
</tr>
<tr>
<td>400</td>
<td>0.539</td>
</tr>
</tbody>
</table>

Table 5.8: Overall reliability ratio, $\eta$, for fly heights 100m - 500m for sensor noise level of 0.2, 0.1, 0.05 and 0.001 nT.

<table>
<thead>
<tr>
<th>heights</th>
<th>$\sigma_v = 0.2$</th>
<th>$\sigma_v = 0.1$</th>
<th>$\sigma_v = 0.05$</th>
<th>$\sigma_v = 0.001$</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0.1056</td>
<td>0.0840</td>
<td>0.0942</td>
<td>0.0025</td>
</tr>
<tr>
<td>200</td>
<td>0.1137</td>
<td>0.0970</td>
<td>0.0291</td>
<td>0.0034</td>
</tr>
<tr>
<td>300</td>
<td>0.1205</td>
<td>0.1046</td>
<td>0.0836</td>
<td>0.0047</td>
</tr>
<tr>
<td>400</td>
<td>0.1262</td>
<td>0.1108</td>
<td>0.0936</td>
<td>0.0065</td>
</tr>
<tr>
<td>500</td>
<td>0.1309</td>
<td>0.1164</td>
<td>0.1003</td>
<td>0.0092</td>
</tr>
</tbody>
</table>

Measurement height and sensor noise

The model in Section 5.6 is used to demonstrate the effect of measurement height on anomalies and various sensor noise levels.

The prior susceptibility model is forward modelled with added noise of 0.1 nT. Figure 5.15 shows the anomalies for heights from 100m to 500m at 100m intervals. The amplitude of the anomalies are largest for the lower flight heights. The improvements in reliabilities for estimations based on measurements at different heights and with sensor noise levels of 0.2, 0.1, 0.05, 0.001 nT are listed in table 5.8.

First, note that according to conventional geophysics methods [73, 25], the flight height should have no effect on how well the parameters can be estimated. This is because of a basic fact about potential fields in free space: if they are known at one height they can be determined exactly at any other height (assuming only free space exists between the two lines). Thus, the measurements taken at 100m, for example, can be "upward continued" to find what the measurements would have been at 500m. Table 5.8, however, shows clearly that the flight height does make a difference; for sensor noise level of, $\sigma_v = 0.2$ nT (left column) the improvements in reliabilities decreases from 0.1056 at 100m height to 0.1309 at 500m height. It is important to understand why.

First of all, the potential field theory assumes that measurements are taken without
any noise at all; the classical potential theory assumes perfect measurements. Second, it assumes that the measurements are continuous, i.e., not sampled in space. This ignores the fact that noise is important, since it limits what can be estimated accurately from a given measurement. Indeed, MMSE can be used to construct a form of continuation that is practical in the presence of noise.

Table 5.8 also shows the same results when the measured noise is very small. The right column in the table shows the improvements in reliabilities for various heights with very low sensor noise, $\sigma_v = 0.001$ nT. In this case the differences between different flight heights is minimal, i.e., the low noise case the standard potential field assumptions hold.

The MMSE method is used to estimate the posterior model using the measurements from multiple flight heights. By combining the A-matrices for 100m and 300m measurement heights to estimate the model, $\eta = 0.1017$ is reported (assumed sensor noise of 0.2 nT). Combining three A-matrices (100/300/500m) the improvement ratio is calculated to $\eta = 0.1000$. This demonstrates an important concept: the estimation is improved by using more than one measurement height. Even the estimation based on the measurements with best quality (at 100m) will be improved by adding measurements taken at higher altitudes. When planning aeromagnetic surveys, measuring at various flight heights should be considered.
Figure 5.16: a) Overall reliability ratio, $\eta$, for fly heights 100m - 500m, sensor noise, $\sigma_v = 0.2$ nT, b) overall reliability ratio, $\eta$, based on 200m fly height and sensor noise of 0.2, 0.1, 0.05, 0.001 nT.
Table 5.9: Prior and estimated reliabilities for each region ($\sigma_i$), (in $10^{-3}$ SI – units) based on various sensor noise levels ($\sigma_v$) and the overall reliability ratio, $\eta$, for the three noise levels (example 1).

<table>
<thead>
<tr>
<th>region</th>
<th>$\sigma_i$</th>
<th>$\sigma_v = 1.0$</th>
<th>$\sigma_v = 0.5$</th>
<th>$\sigma_v = 0.2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.20</td>
<td>0.16</td>
<td>0.14</td>
<td>0.11</td>
</tr>
<tr>
<td>2</td>
<td>0.30</td>
<td>0.27</td>
<td>0.25</td>
<td>0.13</td>
</tr>
<tr>
<td>3</td>
<td>0.40</td>
<td>0.13</td>
<td>0.10</td>
<td>0.08</td>
</tr>
<tr>
<td>4</td>
<td>0.50</td>
<td>0.17</td>
<td>0.14</td>
<td>0.11</td>
</tr>
<tr>
<td>5</td>
<td>0.60</td>
<td>0.16</td>
<td>0.14</td>
<td>0.11</td>
</tr>
<tr>
<td>6</td>
<td>1.00</td>
<td>0.19</td>
<td>0.15</td>
<td>0.12</td>
</tr>
<tr>
<td>$\eta$</td>
<td></td>
<td>0.28</td>
<td>0.23</td>
<td>0.19</td>
</tr>
</tbody>
</table>

Estimation of variance before measurements

As proven in Section 3.6, the variance of the estimated variables (susceptibilities) can be calculated without knowing any measurements.

$$\Sigma_{est} = \Sigma_{(x|y)} = \Sigma_x - \Sigma_x A^T (A\Sigma_x A^T + \Sigma_v)^{-1} A\Sigma_x$$ (5.3)

The estimated variance $\Sigma_{est}$ can be calculated knowing $A$, $\Sigma_x$, and $\Sigma_v$. This enables the evaluation of the contribution of magnetic measurements in improving a geological model; it may even be decided that magnetic measurements are irrelevant in a specific situation because it could supply little additional information.

This capability of the MMSE estimation method is discussed by using the prior model in Section 5.4. The prior susceptibilities are assigned various reliabilities, and different sensor noise levels are assumed in two examples.

In example 1 it is assumed a large uncertainty in the prior variables (susceptibilities in $10^{-3}$ SI – units):

$$\sigma_i = \{0.20, 0.30, 0.40, 0.50, 0.60, 1.00\}$$

for each of the 6 regions. In example 2, low uncertainties in the prior variables are assumed:

$$\sigma_i = \{0.020, 0.030, 0.040, 0.050, 0.060, 0.100\}$$

Table 5.9 and 5.10 shows the prior reliabilities and the estimated reliabilities with sensor noise of $\sigma_v = 1.0, 0.5, 0.2$ nT for the two examples. The overall improvements in reliabilities for the two examples are also listed. Both examples show significant improvements in the reliabilities for the estimated parameters. The best estimation is of course when the prior reliabilities are low, combined with low sensor noise. The conclusion can be made that the estimation is improved by lower sensor noise, and the overall improvement ($\eta$) is significant when the prior uncertainties are big.
Table 5.10: Prior and estimated reliabilities for each region ($\sigma_i$), (in $10^{-3}$ SI – units) based on various sensor noise levels ($\sigma_v$) and the overall reliability ratio, $\eta$, for the three noise levels (example 2).

Imagine these examples as real cases: If the geologist claims a very good reliability in the prior model (example 1) and assuming low quality in the magnetic measurements (e.g., 1.0 nT sensor noise), then the overall improvement in reliabilities (using magnetic measurements) will be $\eta = 0.67$, which means a reduction in the uncertainty of 33%. The conclusion might be that magnetic data is of minor importance. If, on the other hand, the prior uncertainties are big (example 2) and the sensor noise is low, e.g., 0.2 nT, the values of the estimation using magnetic measurements is significant with a reduction in the uncertainty by an average of 81% ($\eta = 0.19$). In this case the use of magnetic measurements could provide important information.
Chapter 6

Application to a real data set

6.1 Introduction to the approach

A real case study from the North Sea is selected to test the MMSE method on a geological model interpreted from a 3D seismic data set.

Whilst exploration for oil and gas, typical steps may be:

1. the area is selected based on 2D seismic, gravity and/or magnetic data, and other available geological information from neighbour areas (e.g. well data).
2. the authorities decide which companies are to be awarded licenses which includes responsibility for further mapping of selected areas
3. 3D seismic surveys are performed, followed by detailed interpretation
4. an exploration play model for the area under investigation is generated
5. the locations for a number of exploration wells are decided

In a standard case, magnetic (and gravity) data are used in a forward modelling fashion; based on geometries, and assumed susceptibilities, an acceptable forward fit is achieved by changing the susceptibilities, and sometimes the geometries iteratively.

In the ideal exploration case outlined above, this new inversion method could be applied in activities 1, 3 and 4. The geometries for the model are derived directly from the interpreted 3D seismic. This geometry is fixed throughout the estimation process due to the linear assumption. Prior susceptibilities are specified by a geologist, based on well logs and general knowledge of the area. Forward modelling is performed to ensure that the prior information is realistic, i.e., give a reasonable forward fit compared to the measured anomalies. An important part of this inversion, is the specification of prior reliabilities to the proposed susceptibilities. Based on the prior information, this MMSE method will calculate estimated susceptibilities
and estimated reliabilities. The various steps in the estimation process are as follows (see figure 6.1):

- the various geological units on the interpreted seismic cross-section is polygonised
- the polygons are colour-coded and imported to the MMSE program
- the colour codes of the pixels \((x)\) are mapped \((F)\) to region numbers \((z)\)
- the \(A\)-matrix is calculated according to the geometry of the pixel model

- prior susceptibilities for each region are assigned based on well information and/or general knowledge
- forward modelling is performed, \(i.e.,\) calculating the anomalies given the geometries and prior susceptibilities
- read the aeromagnetic measurements
- assign prior "best guess" reliabilities
- estimate susceptibilities and reliabilities using the MMSE algorithm
- calculate the information measures \((\eta, \tau)\) and generate various plots

The estimated model is then compared to the prior model. A geological discussion of the values of the estimated susceptibilities and the associated reliabilities for the various regions may reveal some interesting hypothesis.

### 6.2 Geological model

A cross-section has been selected in an area where the strike of the main geological structures are perpendicular to the profile, and the extension of these structures, along the strike is according to the assumptions for a 2D modelling, see Section 4.2. The selected cross-section should also avoid noise from magnetic effects caused by wells and oil installations.

Figure 6.2 shows the interpreted seismic cross-section superimposed on the total model. The various geological units are reduced to a polygonal representation. Compared to the seismic cross-section the model is expanded both to the right and at depth. The basement structures, are derived from regional geological studies in the area and added to the model by a geologist. Each polygon is then assigned a unique colour which is transformed to region numbers \((1 \ldots 56)\), shown in the figure. The magnetic measurement profile at the top of the figure is explained in the next section.

The "depth" in the seismic cross-section, \(z\)-axis in the model, is in two-way travel time (milliseconds) and has a depth of \(\sim 5000\) milliseconds (ca. 5km). No depth
Figure 6.1: The diagram shows the various steps in the MMSE estimation process of real data.
Figure 6.2: The interpreted seismic section is expanded to the right and included basement structures mapped in regional studies, total size: ~ 22km long and ~ 7km deep. The derived geological model consists of 56 different regions (colours), and is sampled into 612 x 563 pixels. The aeromagnetic measurement profile is shown above the model.
### Table 6.1: The mapping between region-numbers and geological units.

<table>
<thead>
<tr>
<th>region</th>
<th>geological units</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>water</td>
</tr>
<tr>
<td>2</td>
<td>Nordland</td>
</tr>
<tr>
<td>3</td>
<td>Hordaland</td>
</tr>
<tr>
<td>4</td>
<td>Cretaceous</td>
</tr>
<tr>
<td>5-8</td>
<td>Heather</td>
</tr>
<tr>
<td>9-15</td>
<td>Brent</td>
</tr>
<tr>
<td>16-23</td>
<td>Dunlin</td>
</tr>
<tr>
<td>24-32</td>
<td>Lunde and Statfjord</td>
</tr>
<tr>
<td>33-41</td>
<td>Lomvi</td>
</tr>
<tr>
<td>42-50</td>
<td>Teist</td>
</tr>
<tr>
<td>51-52</td>
<td>Trias and Perm</td>
</tr>
<tr>
<td>53-56</td>
<td>Basement</td>
</tr>
</tbody>
</table>

conversion has been performed, but using two-way travel time as depth is assumed to be accurate to within a 5% deviation. The total “depth” is ~ 7km and the total length of the section is approximately 22km. The allocation of region-numbers to each of the geological layers is shown in table 6.1.

The prior geological model is sub-sampled to a $62 \times 57$ pixel model by selecting every 10th sample in $x$ and $z$ from the original model, see figure 6.3.

## 6.3 Airborne magnetic measurements

The aeromagnetic measurements were performed during 1993 using a Scintrex Cesium Vapour type MAC-3 sensor in a DHC-6 Twin Otter aeroplane. The data are not RTP-corrected (Reduced-to-Pole). For surveys in areas with high inclination angle, like in this area, this processing it not necessary. The forward modelling and estimation is therefore performed with the true inclination angle at the survey area (72°). The height of the survey was 200 meters above sea level. A ground magnetometer (Scintrex MP-3) was operated as a reference for the whole period. Readings from this base magnetometer were used to decide when to fly and what needed to be re-measured. When the diurnal variations were greater than 10 nT per minute, data were rejected. The recorded data set was first processed with a high resolution processing package to remove all diurnal long wave-length noise. Then the data set was processed by a standard processing package for gravity and magnetics followed by a high-resolution processing package which is designed to take advantage from dense sampling and parallel lines in the survey layout to effectively reduce the noise in the data. The main steps in the standard processing were:
Figure 6.3: The sub-sampled geological model consists of 62 rows and 57 columns, each representing 114m (vertically) × 383m (horizontally) and 56 different regions (colours).

- Spike editing
- Removal of IGRF-90 field
- Editing out anomalies caused by cultural phenomena
- A zero order network adjustment of the lines

After the corrections, the accuracy of the measurements is within 0.2 nT (= $\sigma_v$).

The measurements assume the following specifications:

- Inclination = 72°,
- Declination = 7°,
- Azimuth = 90°
- Magnitude of ambient earth field, $H_0 = 52000$ nT

The measurements, $y$, are taken at a height of 200m, starting at $X = 3671$m and going to $X = 21890$m. The original spacing was on average 53m, and has been sub-sampled to a \(~106$m spacing. Figure 6.2 shows the selected magnetic anomaly profile.

### 6.4 Prior susceptibilities and reliabilities

After the geometry of the model has been defined, the geologists suggest the prior susceptibilities for each region by firstly defining the stratigraphy and then assigning
susceptibilities based on this. The susceptibilities for each rock type are determined using well measurements and susceptibility tables (e.g., table A.1).

To ensure that the susceptibility values for a rock type with 98% probability lies within the range $[a, b]$, it is reasonable to propose:

$$
\bar{\chi} = \frac{(a + b)}{2}
$$

$$
\Sigma = \frac{(b - a)}{4},
$$

i.e., $[a, b]$ gives the $2\sigma$-interval.

These formulas are used to define the prior susceptibilities and reliabilities for each rock type in the prior geological model. This information is listed in table 6.2 where the mean susceptibilities ($\bar{\chi}$) and reliabilities (standard deviation, $\sigma$) for each rock type are specified. All the values are checked by the geologist to lie within a reasonable interval. The assigned susceptibilities in this area are:

- sandstones: $0.2 - 0.5 \times 10^{-3}$ SI - units
- claystones/shales: $0.5 - 1.0 \times 10^{-3}$ SI - units
- sand in Trias: $\sim 1.8 \times 10^{-3}$ SI - units
- basement (metamorphic/igneous rocks): $\sim 8.0 \times 10^{-3}$ SI - units

The range of the susceptibilities in sandstones and shales reflects the sand/shale ratio of each geologic interval. Higher susceptibilities are expected in the Trias/Permian succession due to a commonly occurring coating with iron oxide on the grain surfaces.

## 6.5 Forward modelling

The remanent magnetism is negligible in the uppermost layers (sandstones) and very small in the middle/lower layers. The basement may contain some remanent magnetism, but it is a good approximation to assume no remanent magnetism exists; the assumption of linearity between the susceptibilities and the magnetic measurements is assumed to hold.

Compared to the geological model shown in figure 6.2, the model used in the forward modelling and inversion is expanded 5 columns both to the left and to the right, i.e., 1915m on each side; to reduce the edge effects in the calculated anomalies (see Section 4.6).

Each geological unit is then assigned a susceptibility according to table 6.2, as shown in figure 6.4. The forward modelling is then performed and the resulting anomalies are shown (blue) in figure 6.4 including the measured anomalies (red). In general
there is an acceptable fit between the aeromagnetic measurements and the forward calculated anomalies, however, one major anomaly and a couple of minor ones are present.

In the MMSE estimation of susceptibilities, which is based on the prior susceptibilities combined with the aeromagnetic measurements, the misfits will be corrected and the resulting geological model discussed.

## 6.6 Estimation of magnetic properties

The MMSE estimation of magnetic properties is then performed using the following information:

- the geometries of the prior geological model (figure 6.4)
- assumed susceptibilities and reliabilities (table 6.2)
- the aeromagnetic measurements

The mismatch between the measured and forward modelled anomalies (figure 6.4) is now used in the estimation of susceptibilities. In the first estimation, the Nordland and Hordaland units are assumed to be homogeneous with respect to susceptibilities (each unit is represented by one region). Table 6.2 also shows the prior reliabilities, $\sigma_i$, assigned to each region (geological unit).
Figure 6.4: The prior geological model with assigned susceptibilities (in $10^{-3}$ SI units); the forward modelled anomalies (in blue) and the aeromagnetic measurements (in red).
Figure 6.5 b shows the estimated susceptibility model, values larger than $2 \times 10^{-3}$ SI — units and negative values are colour coded red. Figure 6.5 a shows the forward calculated anomalies based on the estimated model (green), the measured anomalies (red) and the prior forward modelled anomalies (blue). The forward fit (between prior and estimated model) is calculated to $\tau = 0.01$, which indicates a good fit. The overall improvement in reliabilities is calculated to $\tau = 0.53$, which means an average improvement of reliabilities of 47%. Figure 6.5 c shows the difference (in absolute values, $\zeta$) between prior and estimated susceptibilities for each region. We notice that the estimated susceptibilities in many regions have been changed substantially, in the range $0.20 - 1.0 \times 10^{-3}$ SI — units, and even more for the basement ($4.5 \times 10^{-3}$ SI — units). Figure 6.5 d shows the improvement in reliabilities for each region, $\lambda$, which lies in the interval $0.3 - 1.0$. For the basement, the Nordland and Hordaland units, $\lambda \sim 0.3 - 0.5$ indicating an improvement in the range of 50 – 70%. The overall improvement and the reliabilities for each unit shows an acceptable estimation.

Studying the numerical results of the estimation listed in tables A.2 and A.3 shows that some of the estimated susceptibilities are negative (regions 24, 31, 47, 50). This does not make sense geologically and the challenge is to develop a theory of what has been incorrectly modelled. The susceptibility model consists of the geometries of the geological units and their assigned susceptibilities. Possible errors might be:

1. incorrect geometries of the rock units
2. incorrect prior susceptibilities for the units
3. a region is supposed to be homogenous with respect to susceptibilities; does this assumption hold for all regions?

(1) There is no reason to suggest wrong geometries with respect to the stratigraphic units. (2) The assumed prior susceptibilities are in general regarded to be realistic, so the homogenous assumption (3) has to be focused.

The MMSE estimation will always seek a good forward fit, and by achieving this it may be "forced" to change the susceptibilities of some regions to unacceptable values. In this case the mismatches in the anomalies have relatively short wavelengths, which indicates incorrect susceptibilities somewhere in the uppermost layers, e.g., the Nordland and Hordaland units. These units are assumed to be homogenous regions along their 22km length. If the misfit between the prior forward anomalies and the magnetic measurements are caused by heterogeneities, with respect to magnetic properties in these layers, this simply cannot be modelled with these geometries. Thus, the MMSE estimation has to change some smaller regions in the upper part of the model to achieve a good forward fit, resulting estimated incorrect susceptibilities. The conclusion at this point is to test the hypothesis that the Nordland and Hordaland units are heterogeneous with respect to their magnetic properties.

Thus, prior to the next estimation, the Nordland and Hordaland units were divided into 17 sub-regions each, by assigning 4 and 4 columns to separate sub regions.
Figure 6.5: a) The measured (red), the forward calculated prior (blue) and the forward calculated estimated (green) anomalies, b) the estimated susceptibilities, c) the difference between prior and estimated susceptibilities ($\xi$), d) the reliability ratio ($\lambda$) for each region, for the homogenous Nordland and Hordaland region case. ($\times 10^{-3}$ SI – units in b-c.)
The first sub-region (to the left) in the Nordland rock unit constitute region number 2. The successive 16 sub-regions are assigned region numbers 57 - 72. The first sub region (to the left) in the Hordaland rock unit constitute region number 3, and the successive 16 sub-regions are assigned region numbers 73 - 88. The prior susceptibilities and reliabilities for all the sub-regions are assigned the same values as for the previous case, i.e., sub-regions 2 and 57 - 72 are assigned susceptibilities $0.5 \times 10^{-3}$ SI - units, sub-regions 3 and 73 - 88 are assigned susceptibilities $1.0 \times 10^{-3}$ SI - units.

A new estimation is then performed given this modified model geometry. The results again shows a very good fit between the measured anomalies and the forward calculated estimated model ($\tau = 0.005$), see figure 6.6 a. The overall improvement in reliabilities is, $\eta = 0.62$, which indicates that by using the aeromagnetic measurements, the uncertainties in the susceptibilities are reduced by an average of 38%, which is acceptable. The estimated susceptibilities, see figure 6.6 b, seems very reasonable, and no negative susceptibilities occur. Figure 6.6 c shows the difference (in absolute values, $\xi$) between prior and estimated susceptibilities for each region. We notice the following:

- the susceptibilities of the basement were altered by approximately $1.0 \times 10^{-3}$ SI - units, which is acceptable
- the sub-regions within the Nordland unit have changed susceptibilities slightly, within the range $0 - 0.12 \times 10^{-3}$ SI - units
- some of the sub-regions in the Hordaland unit were substantially altered

Figure 6.6 d shows the improvement in reliabilities, $\lambda$, for each pixel. The improvement factor of the basement, $\lambda \sim 0.3 - 0.9$, indicates an improvement in reliabilities in the range of $10 - 70\%$. The improvement factor of the Nordland unit, $\lambda \sim 0.3 - 0.5$, indicates an improvement in reliabilities of $50 - 70\%$, and for the Hordaland unit $\lambda \sim 0.4 - 0.6$, indicating an improvement in the range $40 - 60\%$. All the numerical results from this estimation are listed in tables A.4 and A.5.

The estimation has resulted in a new susceptibility model with a relatively good improvement in reliabilities. All the estimated susceptibilities are acceptable and substantial changes have been made to some of the regions in the Nordland and especially the Hordaland units. This result may be very interesting geologically and is discussed in the next section.

6.7 Discussion

The first estimation, based on the Nordland and Hordaland units being treated as continuous and homogeneous regions, shows that a theoretically good estimation ($\eta = 0.53, \tau = 0.01$), does not necessarily mean a good estimation in a geological
Figure 6.6: a) The measured (red), the forward calculated prior (blue) and the forward calculated estimated (green) anomalies, b) the estimated susceptibilities, c) the difference between prior and estimated susceptibilities ($\xi$), d) the reliability ratio ($\lambda$) for each region, for the heterogeneous Nordland and Hordaland sub region case. ($10^{-3}$ SI – units in b-c.)
sense. The geometries of this case forces the MMSE estimation method to estimate susceptibilities which are obviously wrong (substantial negative susceptibilities).

The next estimation; dividing the Nordland and Hordaland units into 17 sub regions each, maintaining the same prior susceptibilities, gave much better results in a geological sense. By studying figure 6.7, it can be noticed that some sub regions in the Hordaland unit have changed their susceptibilities significantly. The overall improvement in reliabilities were, however, slightly poorer compared to the first case, but still a significant improvement in the overall reliability is achieved. From an interpretation point of view, the changes in susceptibilities are of course more important than the improvement ratio ($\theta$).

The long-wave information from the basement is adjusted within the limits of variations in susceptibilities specified a priori (regions 53 - 54). There are also some adjustments in the regions 43 - 49, 21, 22 and 29, which indicates that the prior susceptibilities, and/or the geological model are slightly incorrect. The main anomaly mismatches have short wavelengths, and are therefore mainly adjusted in the Hordaland unit (regions 74, 77-79, 81-83 and 86). An interesting observation is also that the change in susceptibilities is substantially less in the Nordland unit compared to the Hordaland unit, even though the prior reliabilities are almost the same (0.3 and 0.4$\times$ 10$^{-3}$ SI - units).

In spite of the fact that the Hordaland unit appears as a homogenous layer on the seismic cross-section, the MMSE estimation shows the largest changes immediately above the culmination of the main structural features (regions 81 and 83), thus indicating that the structural features, or processes associated with these features, are continuing upwards into the Hordaland unit, without being detected on the seismic profile. This shows the strength and the usefulness of applying such an independent magnetic modelling in conjunction with traditional seismic interpretation.
Chapter 7

Concluding remarks

In this work the focus has been placed on potential applications of the proposed inversion method in a hydrocarbon exploration context.

The improved quality of modern high resolution aeromagnetic data enables the detection of even subtle variations in the magnetic properties encountered within sedimentary basins. However, the need for putting constraints to the inversion of aeromagnetic data still remains due to the non-uniqueness in the forward modelling, i.e., many subsurface susceptibility models may cause the same aeromagnetic anomalies. Many existing inversion methods “solve” the ambiguity problem by constraining the solutions to simple geometrical models. The present approach enables the implementation of detailed prior information, and utilises the information inherent in the magnetic measurements to improve the prior geological susceptibility model. The proposed inversion method is aiming at being a test-tool for exploiting the magnetic anomalies encountered in the sedimentary basins as a complementary tool for hydrocarbon exploration.

The model is based on the linear relationship that exists between the geological susceptibility model (represented by the A-matrix) and the aeromagnetic measurements. The prior susceptibilities and the sensor noise are assumed to have a Gaussian distribution and the geometry of the model is fixed during the inversion process. The prior susceptibilities and associated uncertainties are assigned based on information from borehole cores or general knowledge about the geology in the region. Various kinds of prior information can be implemented in the estimation process, i.e., spatial correlations between pixels; the definition of homogenous regions; and the encoding of susceptibility gradients within regions. Analysis and evaluations of this A-matrix quantifies what information can be expected from aeromagnetic measurements - given several realisations with various geometries and sensor noise.

Testing the present inversion method on synthetic data shows that a good forward fit in the anomalies does not necessarily mean that the estimated susceptibility model is close to the correct model. The reliability of the estimated susceptibilities will always be improved compared to the input data by using the simulated aeromagnetic data, but the improvement factor, $\eta$, depends on both the degree of details in the
prior assumptions and the noise level of the sensor. If the prior susceptibility model is "close" to the correct model, a significant improvement in the reliability of the estimated results, in addition to a good forward fit in the anomalies.

The proposed inversion method is also tested on a real data set from a North Sea oil field. The geometry of the model is derived from a seismic cross-section and expanded downwards based on regional geological studies. The geological model is verified by exploration wells in the area. The geological units in the model have fixed geometries and a constant susceptibility within each region. It is assumed that no correlation exists between the geological units. The assigned prior susceptibilities are based on "normal" values for the various sedimentary rocks and available well core information. The resulting geological model of magnetic susceptibilities is forward modelled and compared to real aeromagnetic measurements covering the same profile. A reasonably good fit is obtained, except for some distinct anomaly mismatches with relatively short wavelengths. From a geological point of view, these mismatches are regarded as revealing some important information: the reasons for the observed mismatches may be (i) incorrect geometrical model and/or (ii) the assumed susceptibilities are erroneous. It is assumed that the geometries of the geological units are correct, so the focus is put on the prior susceptibility input data. It was observed that the estimated susceptibilities for some of the regions were negative, which is obviously wrong. This can be explained by the following: to achieve a good forward fit, the inversion method is "forced" to assign negative values for some of the regions, indicating that local variations in susceptibilities occurs within assumed homogenous layers. The obvious candidates are the two uppermost geological units because short wavelength anomalies must originate from shallow rock bodies. It is therefore assumed that the susceptibility heterogeneities occur within the 20 kilometre long Nordland and Hordaland units. By dividing these units into sub-regions, an improvement of the estimated susceptibility model results. All of the estimated susceptibilities are now within the "normal" range. In addition, some of the sub-regions in the Nordland and Hordaland units are changed significantly; thus confirming the hypothesis of heterogeneities in the prior assumed homogenous geological layers.

Seismic data is the most important information source in hydrocarbon exploration. Further detailing of the geological model derived from the seismic data can only be achieved by drilling wells. By using an independent data set, e.g., aeromagnetic measurements, the prior susceptibility model can be verified by a simple forward modelling. If a mismatch between the forward calculated anomalies and the measured anomalies occur, this indicates that the prior susceptibility model is wrong.

An interesting spin-off from this inversion method is the possibility of estimating the expected improvement in reliability before any aeromagnetic measurements are made. This option can be used to plan aeromagnetic surveys or to evaluate the contribution of existing aeromagnetic measurements given a prior statistical susceptibility model. The method may also be applied in areas with very little prior geological information. If, for example, aeromagnetic data is available and the interpreter is looking for faults in some directions (or other features with a defined
geometry) the use of distance correlation matrices may be applied to detect possible occurrences of such features in the subsurface.

The inversion method presented here, has proven very effective at improving the prior susceptibility model by a user-friendly human interacting process. In a real exploration scenario, this inversion method may be used to tune the geological model by blending the prior information with aeromagnetic measurements in an optimal way, resulting in a more detailed and “correct” geological interpretation but with an increased confidence.

**Future work**

This work has been restricted to 2D modelling, but the method should be expanded to 3D because the 2D assumptions restrict the application of the method. In the 3D model, the pixels will be voxels (volumes), and the size of the model will increase significantly, causing a need for more efficient estimation methods. Such a 3D model, will also create a need for a more efficient programming code, large memory space and computational power.

Implementing methods for more efficient inversion of large scale problems should also be considered. Oldenburg and Li [56] have presented efficient algorithms to solve large scale linear inverse problems by dividing the space into a number of sub-spaces and iteratively inverting these sub-spaces.

This estimation method could be developed for gravimetric data (in 2D), but 3D modelling is vital, because a gravity measurement is significantly influenced by the neighbouring geological units.

A pre-condition for a wide spread use of this novel method, is to develop an user-friendly menu driven interface. The geometries of the geological model should be imported directly from the seismic interpretation workstation and a database with standard susceptibilities for common rock-types should be available.
Bibliography


[58] L.B. Pedersen, 1979, Constrained inversion of potential field data Geophysical Prospecting, 27(4),726–748.


Appendix A

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Table A.2: Prior ($\bar{z}$) and estimated ($\hat{z}$) susceptibilities and reliabilities ($\sigma_i$ and $\hat{\sigma}_i$), and change in susceptibilities and reliabilities, for each region (continues next page).
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Table A.3: Prior ($\tilde{z}$) and estimated ($\hat{z}$) susceptibilities and reliabilities ($\sigma_1$ and $\hat{\sigma}_1$), and change in susceptibilities and reliabilities, for each region (see previous page).
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Table A.4: Prior ($\bar{z}$) and estimated ($\hat{z}$) susceptibilities and reliabilities ($\sigma_i$ and $\hat{\sigma}_i$), and change in susceptibilities and reliabilities, for each region (continues next page).
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Table A.5: Prior ($\bar{z}$) and estimated ($\hat{z}$) susceptibilities and reliabilities ($\sigma_1$ and $\hat{\sigma}_1$), and change in susceptibilities and reliabilities, for each region (see previous page).
Appendix B
List of symbols

$P_{ij}$ a rectangular model divided into $n_z$ times $n_x$ pixels

$n_x$ number of pixels in $x$ direction (horizontally)

$n_z$ number of pixels in $z$ direction (vertically)

$\Delta X$ size of pixel in $x$ direction

$\Delta Z$ size of pixel in $z$ direction

$\Delta Y$ samplings density (in meters)

$\Delta T$ changes in total magnetic field

$H_0$ magnitude of the ambient magnetic field

$A$ the matrix relating variables to measurements

$\tilde{A}$ the A-matrix scaled to give signal-to-noise ratio

$A^{-1}$ inverse of $A$

$A^T$ transpose of $A$

$x$ the true value of the parameter to estimate

$\bar{x}$ the mean value of $x$, i.e., our best prior guess of what the parameters are

$\hat{x}$ estimated value of $x$

$\tilde{x}$ estimation error of $x$, i.e., $\tilde{x} = (x - \hat{x})$

$|\tilde{x}|$ absolute estimation error of $x$, i.e., $|\tilde{x}| = |x - \hat{x}|$

$z$ as $x$, used in the region/gradient models

$\bar{z}$ as $\bar{x}$, used in the region/gradient models

$\hat{z}$ as $\hat{x}$, used in the region/gradient models

$\xi$ absolute values of the difference between prior and estimated susceptibilities, $\xi = |\bar{x}_i - \hat{x}_i|$
\( \bar{v} \) the mean value or bias or offset of the measurement

\( \Sigma_x \) covariance matrix of \( x \), which describes the confidence in the prior guess \( \bar{x} \) for \( x \)

\( \Sigma_{\text{est}} \) covariance matrix of \( \bar{x} \)

\( \Sigma_v \) covariance matrix of \( v \), which describes the variability of the measurements

\( \sigma_i \) prior reliabilities in \( x_i \) or \( z_i \), one standard bar (std.)

\( \hat{\sigma}_i \) estimated reliabilities in \( x_i \) or \( z_i \), one standard bar (std.)

\( \sigma_v \) reliabilities in the measurements, \( \sigma_v = \sqrt{\Sigma_v} \)

\( \rho_{ij} \) the correlation between pixel \( i \) and \( j \)

\( y \) the magnetic sensor measurement vector

\( \bar{y} \) the expected value vector of sensor measurements

\( \kappa \) susceptibility (magnetic property) of rocks

\( \kappa_i \) susceptibilities for pixel/region no. \( i \)

\( \lambda_i \) the improvement in reliability ratio of \( \hat{x}_i \) compared to \( \bar{x}_i \) (also used for regions, \( z \))

\( \eta \) the posterior to prior ratio of reliabilities, RMS deviation

\( \tau \) a measure of mismatch between anomalies

\( \cong \) equal by definition

\( \sim \) similar to

\( |a| \) the absolute value of \( a \)

\( \|a\| \) the norm of \( a \), i.e., \( \sqrt{a^T a} \)

\( \det \Sigma \) the determinant of matrix \( \Sigma \)

\( \text{Tr} \) denotes the trace of a matrix, i.e., the sum of its diagonal elements

\( \text{rank}(a) \) the number of independent rows in matrix \( a \)

\( \text{min}(z) \) the minimum value in vector \( z \)

\( \text{max}(z) \) the maximum value in vector \( z \)
Appendix C
List of definitions

azimuth used to specify flight profile direction, measured clockwise, relative to North

basement the crust of the Earth below sedimentary deposits, extending downwards to the Mohorovicic discontinuity

CDF Cumulative Distribution Function

core the central zone or nucleus of the Earth’s interior

cross-section a vertical cut showing transected geological features

crust the outermost layer or shell of the Earth; the part of the Earth above the Mohorovicic discontinuity

decline the angle between the Earth’s magnetic field and geographic north

emu/cgs the basis is the point magnetic poles, \( \kappa_{emu} = \frac{\kappa_{SL}}{4\pi} \)

fault a discrete surface or zone of discrete surfaces separating two rock masses across which one has slid past the other

\( \gamma \) cgs-unit for measuring magnetic induction, \( 1\gamma = 1nT \)

incline the dip of the Earth’s magnetic field at a specific location

lithological (unit) a defined body of sedimentary, igneous, etc. strata that is distinguished and delimited on the basis of lithic characteristics and stratigraphic position

magnetization magnetic polarisation, dipole moment per unit volume, \( M \)
\[
M = \kappa \cdot H_0
\]

mantle the zone of the Earth below the crust and above the core

mineral a naturally occurring inorganic element or compound

nT unit for measuring magnetic induction, \( 10^{-9} \) T

pixel a fixed size rectangular cell with constant magnetisation, representing parts of the subsurface (in 2D)
Positive definite \( x^T A x \) is positive except at \( x = 0 \)

Q  
Koenigsberger ratio (dimensionless), ratio between remanent and induced magnetisation

RMS  
Root Mean Square

rock  
an aggregate of one or more minerals, \textit{e.g.}, granite

RTP  
Reduction-to-pole, transformation of the anomalies to inclination 90°

sediment  
solid fragmental material that originates from weathering of rocks

SI-units  
the basic unit is the dipole ampere per meter (A/m), \( \kappa_{SI} = 4\pi \kappa_{emu} \)

strike  
the direction or trend taken by a structural surface

susceptibility  
a dimensionless number (\( \kappa \)) characterising the various rock’s abilities to be magnetised within a magnetic field

well  
a bore hole or shaft sunk into the ground for the purpose of obtaining oil and/or gas from the underground source, introducing water or gas under pressure into an underground formation or for mapping the geological layers
Appendix D

Eigenvalues, eigenvectors and SVD

This short introduction to eigenvectors, eigenvalues and singular value decomposition is adapted from [4, 70].

If the matrix $A \in \mathbb{R}^{n \times n}$, then a non zero vector $x \in \mathbb{R}^n$ is called an eigenvector of $A$ if $Ax$ is a scalar multiple of $x$, that is:

$$Ax = \lambda x$$

for some scalar $\lambda$. The scalar $\lambda$ is called the eigenvalue of $A$ and $x$ is said to be an eigenvector corresponding to $\lambda$.

$$\lambda_1 \cdot \lambda_2 \ldots \lambda_n = \det(A)$$

$$\lambda_1 + \lambda_2 + \ldots + \lambda_n = \text{Tr}(A) = \sigma_{11} + \sigma_{22} + \ldots + \sigma_{nn}$$

Let $A \in \mathbb{R}^{m \times n}$ be a matrix, and $U$ and $V$ be two unitary matrices. Then

$$U^TAV = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}$$

where

$$\Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r)$$

and

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0$$

The $\sigma_i$'s are the singular values of $A$, denoted $SVD(A)$, and $r$ is the rank of $A$. If $A$ is a symmetric matrix, then its singular values equal the absolute values of the eigenvalues of $A$. 

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