Introduction to Mathematical Morphology

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Outline

• Introduction
• Mathematical background
• Dilation and erosion
• Opening and closing
• Hit and miss transform, skeletons
• Geodesic dilation, erosion, and reconstruction
• Watershed segmentation
Mathematical Morphology - Introduction

- Based on shapes in the image, not pixel intensities
- Can be viewed as a general image processing framework
  - Various image processing techniques can be implemented by combining only a few simple operations
    - Examples: gradients, distance images, skeletons, noise removal, contrast enhancement, filling
- Typically used before and after an image segmentation
  - Exception: watershed segmentation
- All mathematical morphology operations are based on dilation and erosion
- The image processing toolkit in Matlab includes many mathematical morphology operations
Mathematical background

- In mathematical morphology we regard the pixel intensities as topographical highs
Mathematical background

- Set theory is used to define dilation and erosion
- We can transform an image into a set of points by regarding the image's subgraph
  - The subgraph of an image is defined as the set of points that lies between the graph of the image and above the image plane

\[
SG(f) = \{(x, t) \in \mathbb{Z}^n \times \mathbb{N} | 0 \leq t \leq f(x)\}
\]
Erosion

- Used to reduces objects in the image
- Definition, binary images:
  - The positions where a given structure element fits
    \[ \varepsilon_B(X) = \{ x \mid B_x \subseteq X \} \]
    where \( B_x \) means \( B \) translated with \( x \),
    \( X \) is the image, and
    \( B \) is the structure element
Erosion

- Example, binary image
Erosion

- Definition, grayscale images
  We remember the definition for binary images:
  \[ \varepsilon_B(X) = \{ x \mid B_x \subseteq X \} \]

  Can be rewritten into the intersections of the translated sets \( X_{-b} \):
  \[ \varepsilon_B(X) = \bigcap_{b \in B} X_{-b} \]

  Which can be extended to also include grayscale images:
  \[ \varepsilon_B(f) = \bigwedge_{b \in B} f_{-b} \]
Erosion

- Based on

\[ \varepsilon_B(f) = \bigwedge_{b \in B} f - b \]

we can define an algorithm to find the erosion of image \( f \):

\[ [\varepsilon_B(f)](x) = \min_{b \in B} f(x + b) \]
Erosion

- Example, grayscale image
Dilation

• Used to increase objects in the image
• Definition, binary images:
  – The positions where a given structure element fits
    \[ \delta_B(X) = \{ x \mid B_x \cap X \neq \emptyset \} \]
    where \( B_x \) means \( B \) translated with \( x \),
    \( X \) is the image, and
    \( B \) is the structure element
Dilation

- Example, binary image
Dilation

- Definition, grayscale images
  We remember the definition for binary images:
  \[ \delta_B(X) = \{ x | B_x \cap X \neq \emptyset \} \]
  Can be rewritten into the unions of the translated sets \( X_{-b} \):
  \[ \varepsilon_B(X) = \bigcup_{b \in B} X_{-b} \]
  Which can be extended to also include grayscale images:
  \[ \varepsilon_B(f) = \bigvee_{b \in B} f_{-b} \]
Dilation

- Based on

\[ \varepsilon_B(f) = \bigvee_{b \in B} f_{-b} \]

we can define an algorithm to find the erosion of image \( f \):

\[ [\varepsilon_B(f)](x) = \max_{b \in B} f(x + b) \]
Dilation

- Example, grayscale image
Dilation

- Example 2, grayscale image
Dilation and erosion example
Beucher gradient

- Dilation and erosion can be used to extract edge information from images
  - Example: Beucher gradient

\[ \rho_B = \delta_B - \varepsilon_B \]
Morphological opening

- Used to remove unwanted structures in the image (e.g. noise)
- Morphological opening is simply an erosion followed by a dilation:
  \[ \gamma_B(f) = \delta_B \left[ \varepsilon_B(f) \right] \]
- Binary example:
Morphological opening

- Grayscale example
Morphological closing

- Is used to merge or fill structures in an image
- Morphological closing is dilation followed by erosion:
  \[ \phi_B(f) = \varepsilon_B[\delta_B(f)] \]
- Binary example:
Another example

- How to segment the text from the uneven illumination in the image?
Another example

• How to segment the text from the uneven illumination in the image?
Another example

- How to segment the text from the uneven illumination in the image?
Hit and miss transform

- Used to extract pixels with specific neighbourhood configurations from an image
- Defined only for binary images
- Uses two structure elements $B_1$ and $B_2$ to find a given foreground and background configuration, respectively

\[ HMT_B(X) = \{ x | (B_1)_x \subseteq X, (B_2)_x \subseteq X^C \} \]

- Example:
Thinning

• Used to shrink objects in binary images
• Differs from erosion in that objects are never completely removed
• Successive thinning until stability results in object skeletons
• Thinning is defined as:

$$THIN(X, B) = X \setminus HMT_B(X)$$

• Structure elements typically used in thinning (rotated 90 degrees 3 times to create 8 structure elements):
Skeletons

- Minimal representation of objects in an image while retaining the Euler number of the image
  - The Euler number is the number of objects in an image minus the number of holes in those objects
- As stated earlier, the skeletons of objects in an image is found by successive thinning until stability
Geodesic dilation

- In geodesic dilation the result after dilating the (marker) image $f$ is masked using a mask image $g$

$$\delta_g(f) = \delta(f) \land g$$

Marker

Mask

Result after dilating the marker image

Masked result
Geodesic erosion

- Similar to geodesic dilation, the result after erosion is masked with a mask image $g$

$$\varepsilon_g(f) = \varepsilon(f) \lor g$$
Geodesic dilation and erosion

- Geodesic dilation repeated n times is expressed and given by
  \[ \delta_g^{(n)}(f) = \delta_g \left[ \delta_g^{(n-1)}(f) \right] \]

- Likewise, geodesic erosion repeated n times is expressed and given by
  \[ \varepsilon_g^{(n)}(f) = \varepsilon_g \left[ \varepsilon_g^{(n-1)}(f) \right] \]
Reconstruction

- Reconstruction is based on geodesic dilation and erosion
- Can for instance be used to
  - Find and fill local minima or maxima
  - Suppress local minima or maxima less than a given size
  - Remove noise while not affecting structures of interest
- Reconstruction by dilation is given by:

\[ R_g(f) = \delta_g^{(i)}(f) \]

where

\[ \delta_g^{(i)}(f) = \delta_g^{(i+1)}(f) \]
Reconstruction by dilation, example
Reconstruction by erosion

- Reconstruction by erosion is given by
  \[ R_g^*(f) = \varepsilon_g^{(i)}(f) \]

  where
  \[ \varepsilon_g^{(i)}(f) \varepsilon_g^{(i+1)}(f) \]
Reconstruction by erosion, example
Another reconstruction example

- Using reconstruction to remove noise
Watershed segmentation

• Watershed segmentation divides an image into basins and finds the positions where the basins would meet if they were gradually filled
Watershed segmentation

Problem: too many local minima results in oversegmentation
Watershed segmentation

- Solution: Use a marker to mark wanted minima, and fill the rest of the local minima
Watershed segmentation

- Final segmentation result
References